Symbolic Logic: QL Proofs

David W. Agler

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QL proofs

- We have the language of QL
- There is a tree procedure (checks for validity, tautology, etc.)
- We will create the proof system: it builds on PL proofs.

Universal Elimination ($\forall E$)

Definition (Universal Elimination ($\forall E$))

From any universally quantified wff $(\forall x)\phi(x_1, \ldots x_n)$, the wff $\phi(a_1 \ldots a_n / x_1 \ldots x_n)$ can be derived. $(\forall x)\phi(x_1 \ldots x_n) \vdash \phi(a_1 \ldots a_n / x_1, \ldots x_n)$

Schema

1	$\forall x \phi(x_1 \ldots x_n)$	P
2	$\phi(a_1 \ldots a_n)$	$\forall E$ 1

MWE

1	$(\forall x)Px$	P
2	Pa	$\forall E$ 1

- P1: Everyone is a person.
- C: Therefore, Al is a person. (by $\forall E$).

Tip 1

1. Remove the universal quantifier
2. Consistently replace the variables it binds with any names you want.

1	$(\forall x)Px$	P
2	Pa	$\forall E$ 1
3	Pb	$\forall E$ 1
4	Pc	$\forall E$ 1

Tip 2

If more than one of the same type of variable is bound by the universal quantifier, you must replace variables with names *entirely* and *uniformly*.

1	$(\forall x)Lxx$	P
2	Laa	$\forall E$ 1
3	Lbb	$\forall E$ 1

Mistake

- Everyone loves themselves.
- Therefore, Al loves Bob.

1	$(\forall x)Lxx$	P
2	Lab	$\forall E$ 1, NO!

Exercise

Ex.8.83, 1,3

Existential Introduction

Definition (Existential Introduction ($\exists I$))

From $\phi(a_n)$, an existentially quantified wff $(\exists x)\phi(x_i/a_n)$ can be derived $\phi(a_n) \vdash (\exists x)\phi(x_i/a_n)$, provided: (1) $i \leq n$ and (2) the variable x_i is not in the original $\phi(a_n)$, viz., x_i is not a variable already bound by a quantifier.

Schema

1	$\phi(a_n)$	P
2	$(\exists x)\phi(x_i/a_n)$	$\exists I$

Uniformly replace at least one name a with an existentially quantified variable x

MWE

1	Za	P
2	$(\exists x)Zx$	$\exists I/1$

P1	Al is a zombie.	Prem
C	Therefore, someone is a zombie.	$\exists I$, from 1

Clarification 1

You can replace more than one name but it needs to be the same name with the same variable.

- Bob loves himself.
- Therefore, someone loves themselves.

1	Lbb	P
2	$(\exists x)Lxx$	$\exists I$ 1
3	$(\exists x)Lbx$	$\exists I$ 1
4	$(\exists x)Lxb$	$\exists I$ 1

Clarification 2

You cannot reason to $(\exists x)Lxx$ from Lab

1	Lab	P
2	$(\exists x)Lax$	$\exists I$ 1
3	$(\exists x)Lxb$	$\exists I$ 1
4	$(\exists x)Lxx$	$\exists I$ 1. NO!

Clarification 3

You can use $\exists I$ on any wff with a name, even an existentially quantified wff.

1	Lab	P
2	$(\exists x)Lax$	$\exists I$ 1
3	$(\exists y)(\exists x)Lyx$	$\exists I$ 2

Notice that the quantifier $(\exists y)$ is added to the left of the wff at line 2.

Exercise

Ex.8.84, 1,5

Universal Introduction

Definition (Universal Introduction ($\forall I$))

Let α be any name. From a wff $\phi(\alpha_1 \dots \alpha_n)$, a universally quantified wff $(\forall x)\phi(x_1 \dots x_n/\alpha_1 \dots a_n)$ can be derived where (1) the name α does not occur in a wff as premise or as an assumption in an open subproof, (2) the name α does not occur in the derived wff $(\forall x)\phi(x_1 \dots x_n/\alpha_1 \dots \alpha_n)$, and (3) where x is not a variable in ϕ.
$\phi(\alpha_1 \dots \alpha_n) \vdash (\forall x)\phi(x_1 \dots x_n/\alpha_1 \dots \alpha_n)$

Restriction 1

The name α does not occur in a wff as a premise or an assumption in an open subproof.

- P1: John won the lottery.
- C: Therefore, everyone won the lottery. No!

1	Let's assume John won the lottery.	A
2	Therefore, everyone won the lottery.	$\forall I$ 1, *Incorrect!*

Example

1	Wj	P
2	$(\forall x)Wx$	$\forall I$ 1. *No!*

1	Wj	A
2	$(\forall x)Wx$	$\forall I$ 1, *Incorrect!*

Restriction 2

If $\forall I$ is used on a wff containing more than one name of the same type (e.g., two *a*s), each name must completely replaced by universally quantified variables. The name α cannot not occur in the derived wff $(\forall x)\phi(x_1 \ldots x_n/\alpha_1 \ldots \alpha_n)$

1. If using $\forall I$ on Lbb, replace both *b*s
2. $Lbb \nvdash (\forall x)Lbx$
3. $Lbb \vdash (\forall x)Lxx$

Restriction 2 - Illustrated

1	Laa	?
2	$(\forall x)Lxx$	1, $\forall I$. OK
3	$(\forall x)Lax$	1, $\forall I$. NO!

Restriction 3

The third restriction says you can't replace names with a variable x if x is already in the wff.

1 $(\exists x)Lax$?

2 $(\forall x)(\exists x)Lxx$ $\forall I$. NO!

Restriction is to ensure that it is clear which quantifier binds which variable.

MWE

1	$(\forall x)Lxx$	P
2	Laa	$\forall E$ 1
3	$(\forall y)Lyy$	$\forall I$ 2

1	Everyone loves themselves	P
2	Al loves Al	$\forall E$ 1
3	Everyone loves themselves	$\forall I$ 2

Intuitions

Intuitive ideas

1. If you can show something is true about an arbitrary object a, then this is true of every object.
2. If Fa is the case and a could have been any object, then $(\forall x)Fx$
3. I could shown the same thing with Fb, Fc, Fd, etc.

Illustration

1. Suppose I pick a triangle and there is nothing special about the triangle (we don't care about its length, height, etc.).
2. I show that the triangle's angles equal 180.
3. Therefore, all triangles angles equal 180.

I picked the triangle, but if you give me a different triangle, the same proof would apply since there is nothing relevantly different (to the proof) about the triangles.

Example

1	Raa	A
2	Raa	R 1
3	$Raa \rightarrow Raa$	$\rightarrow I$ 1-2
4	$(\forall x)(Rxx \rightarrow Raa)$	$\forall I$ 3

I could prove line 3 with different names, e.g., $Rbb \rightarrow Rbb$

Exercise

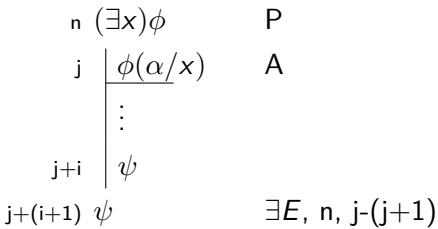
Ex.8.85, 2, 3

Existential Elimination

Definition (Existential Elimination ($\exists E$))

Let α be any name and x be any variable. From an existentially quantified wff $(\exists x)\phi$, a wff ψ can be derived from $(\exists x)\phi$ and a subproof that begins with $\phi(\alpha_n/x_n)$ and ends with ψ, provided (1) the name α does not occur in any premise or in an active proof (or subproof) prior to its arbitrary introduction in the assumption $\phi(\alpha_n/x_n)$ and (2) the name α does not occur in ψ.

Schema



1. $\exists E$ is an elimination rule
2. requires a subproof
3. certain restrictions on using $\exists E$

MWE

1	$(\exists x)Px$	P
2	$\vdash Pb$	A

MWE

1	$(\exists x)Px$	P
2	$\vert\ Pb$	A
3	$\vert\ (\exists y)Py$	$2\exists I$

MWE

1	$(\exists x)Px$	P
2	$\vert\ Pb$	A
3	$\vert\ (\exists y)Py$	$2\exists I$
4	$(\exists y)Py$	1, 2-3$\exists E$

Notice that we cite the existentially quantified wff and the entire subproof.

Mistake 1

1	$(\exists x)Px$	P
2	Pa	$\exists E$, No!

No subproof.

Mistake 2

1	$(\exists x)Mx$	P
2	$\vdash Mb$	A
3	Mb	1-2$\exists E$. No!

Violates the restriction that you cannot derive a wff containing a name b you've assumed.

Mistake 3

1	$Mb \rightarrow Nc$	P
2	$(\exists x)Mx$	P
3	$\vert\ Mb$	A
4	$\vert\ Nc$	R, 1
5	Nc	$\exists E$ 2, 3-4, *Incorrect!*

No. The name you assume cannot occur in any premise or in an active proof (or subproof) prior to its arbitrary introduction in the assumption $\phi(\alpha_n/x_n)$.

Exercise

Ex.8.86, 1,2

QN

Definition (Quantifier Negation (QN))

$\neg(\forall x)\phi \dashv\vdash (\exists x)\neg\phi$

$\neg(\exists x)\phi \dashv\vdash (\forall x)\neg\phi$

MWE 1

1	$\neg(\forall x)Px$	P
2	$(\exists x)\neg Px$	QN 1
3	$\neg(\forall x)Px$	QN 2

- P1: It is not the case that everything is P.
- C: Therefore, something is not P.

MWE 2

1	$\neg(\exists x)Px$	P
2	$(\forall x)\neg Px$	QN 1

- P1: It is not the case that something is P.
- C: Therefore, everything is not P.

Clarification

QN can be applied to subformula.

1 $\neg(\forall x)\neg(\exists y)Rxy$ P

2 $(\exists x)\neg\neg(\exists y)Rxy$ QN 1

Let's apply it to the antecedent of a conditional:

1 $\neg(\forall x)Px \rightarrow Pb$ P

2 $(\exists x)\neg Px \rightarrow Pb$ QN 1

Exercise

Ex.8.87, 1, 2