

Symbolic Logic: PL Proofs

David W. Agler

October 26, 2026

Two notions of entailment

- ▶ Intuitively, a good argument is one where the conclusion follows from the premises or assumptions.
- ▶ A set of wffs can be said to entail a wff in two different ways

Two notions of entailment

First, there is *semantic entailment* or model-theoretic entailment.

Definition (Semantic entailment)

A wff φ is **semantically entailed** by a set of wffs Γ iff there is no interpretation where the members of Γ are T and φ is F.

Second, there is *syntactic entailment* or proof-theoretic entailment.

Definition (Syntactic entailment)

A wff φ is **syntactically entailed** by a set of wffs Γ iff there is a proof (or derivation) of that wff.

What is a derivation (proof)?

What is a **derivation** (or proof)?

Definition (derivation of φ in PD)

A **derivation** (proof) of φ from a set of wffs Γ is a finite string of wffs from a set Γ of PL wffs where (i) the last wff in the string is φ and (ii) each wff's place in the string is justified.

Justified?

What does it mean for a wff in the string to be justified?

1. wff is in Γ (so it is a premise)
2. wff is permitted by the use of the deductive apparatus (rules or reason, inference rules)

what is a deductive apparatus?

What is the deductive apparatus?

Definition (deductive apparatus)

A deductive apparatus for PL is a set of rules of that state which wffs can be written in a proof.

More intuitively:

- ▶ P1: If Tek is happy, then he is smiling.
- ▶ P2: Tek is happy.
- ▶ C: Tek is smiling

Rule: whenever you have $\phi \rightarrow \psi$ and ϕ , then you can write ψ .

syntactic consequence

Definition (syntactic consequence (entailment))

A formula φ is a syntactic consequence in PD of a set Γ of PL wffs if and only if there is a derivation of φ from Γ .

To express that φ is a syntactic consequence of Γ , we write $\Gamma \vdash \varphi$.

- ▶ $P, M, R \vdash Q$
- ▶ What is said here is that Q is a syntactic consequence (is syntactically entailed by P, M, R)
- ▶ In other words, there is a proof that begins with P, M, R and ends with Q .

Derivation Setup (Fitch-Style)

Now that we know what a proof is, we now will look at how to set up a proof.

- ▶ Consider the following: $P \wedge R, Y \rightarrow R \vdash Z$.
- ▶ This set of symbols says that Z is a syntactic consequence of $P \wedge R, Y \rightarrow R$.
- ▶ it says that there is a derivation from $P \wedge R, Y \rightarrow R$ to Z .

Step 1

To show that there is a derivation, we need to *set up the proof*.

Derivation Setup (Fitch-Style)

The initial setup involves three columns:

1. for numbering the premises,
2. writing (stacking) the propositions,
3. justification of propositions and indicating the goal proposition (or conclusion)

How to set up a proof

$$P \wedge R, R \rightarrow Z \vdash Z$$

$$1 \quad P \wedge R \quad P$$

$$2 \quad R \rightarrow Z \quad P, Z$$

- ▶ $P \wedge R$ and $Y \rightarrow R$ are premises (and we use P)
- ▶ The conclusion (goal proposition) is Z

Exercise

1. Ex.5.52, p.187, 1
2. Ex.5.52, p.187, 6

The Deductive Apparatus

We need to create the deductive apparatus.

- ▶ Remember the deductive apparatus is just a set of rules that allow us to write new wffs in the proof.
- ▶ In other words, they are rules that allow us to go from the premises to the conclusion.

The Deductive Apparatus

- ▶ We'll use a system of natural deduction (sometimes abbreviated as *PD*)
- ▶ The particular rules of PD will be called the “derivation rules”.

The Deductive Apparatus: Intelim System

PD will consist of two main types of derivation rules:

1. **introduction rules** (these introduce a wffs of a certain type into a proof)
2. **elimination rules** (these begin from wffs of a certain type in a proof).

PD Conjunction Introduction

Definition (Conjunction Introduction $\wedge I$)

From P and Q , we can derive $P \wedge Q$. Also, from P and Q , we can derive $Q \wedge P$.

More abstractly:

1. $\phi, \psi \vdash \phi \wedge \psi$
2. $\phi, \psi \vdash \psi \wedge \phi$

Conjunction Introduction

The idea here is we can reason from two separate wffs to their conjunction.

1. “John is a man”
2. “John is a banker”
3. *Therefore*, “John is a man and a banker.”

Example

$P, R, W \vdash P \wedge (W \wedge R)$ First, setup up the proof.

1 P P

2 R P

3 W P

Example

$P, R, W \vdash P \wedge (W \wedge R)$ Next, let's use $\wedge I$:

1	P	P
2	R	P
3	W	P
4	$\boxed{W \wedge R}$	$\wedge I\ 2, 3$

Example

$$P, R, W \vdash P \wedge (W \wedge R)$$

Now get the conclusion with $\wedge I$ again:

1	P	P
2	R	P
3	W	P
4	$W \wedge R$	$\wedge I$ 2, 3
5	$P \wedge (W \wedge R)$	$\wedge I$ 1, 4

Conjunction Elimination

Definition (Conjunction Elimination $\wedge E$)

From $P \wedge Q$, we can derive P or Q .

More abstractly:

1. $\phi \wedge \psi \vdash \phi, \psi$

Conjunction Elimination

The idea here is if we have an AND sentence, we can reason to either side.

1. Tek is a criminal and a jerk.
2. *Therefore*, John is a criminal.
3. *Therefore*, John is a jerk.

Example

$$P \wedge (Q \wedge W) \vdash Q$$

Setup

$$_1 \quad P \wedge (Q \wedge W) \quad P, Q$$

Example

$$P \wedge (Q \wedge W) \vdash Q$$

Use $\wedge E$ on line 1

- | | | |
|---|-------------------------|---------------|
| 1 | $P \wedge (Q \wedge W)$ | P, Q |
| 2 | $Q \wedge W$ | $\wedge E, 1$ |

Use $\wedge E$ on line 2

- | | | |
|---|-------------------------|---------------|
| 1 | $P \wedge (Q \wedge W)$ | P, Q |
| 2 | $Q \wedge W$ | $\wedge E, 1$ |
| 3 | Q | $\wedge E, 2$ |

Exercise

1. Ex.5.53, p.193, 1

Exercise

1. Ex.5.53, p.193, 1
2. Ex.5.53, p.193, 2

Exercise

1. Ex.5.53, p.193, 1
2. Ex.5.53, p.193, 2
3. Ex.5.53, p.193, 5

Conditional elimination

Definition (Conditional Elimination ($\rightarrow E$))

$\phi \rightarrow \psi, \phi \vdash \psi$. From $\phi \rightarrow \psi$ and ϕ , the wff ψ can be derived.

Conditional elimination in English

1. If Tek is guilty, then he deserves punishment.
2. Tek is guilty.
3. Therefore, he deserves punishment (by 1, 2 and $\rightarrow E$)

Simple Example

1	$P \rightarrow Q$	P
2	P	P, Q
3	Q	$\rightarrow E$ 1, 2

More complex example

- | | | |
|---|-----------------------------------|----------------------|
| 1 | $(A \wedge B) \rightarrow \neg Q$ | P |
| 2 | $(A \wedge B)$ | P, Q |
| 3 | $\neg Q$ | $\rightarrow E$ 1, 2 |

Even more complex example

1	$\neg(A \wedge B)$	P
2	W	P
3	$\neg(A \wedge B) \rightarrow \neg\neg W$	P, Q
4	$\neg\neg W$	$\rightarrow E$ 1, 3

Common Mistake

$$\begin{array}{l} 1 \quad P \rightarrow Q \quad P \\ 2 \quad Q \quad \quad \rightarrow E \ 1 \end{array}$$

Wrong! You need P on a separate line.

Common Mistake

1	$P \rightarrow Q$	P
2	P	P
3	Q	$\rightarrow E\ 1$

Still wrong! You need to cite line 2 (P) in the justification at line 3.

Exercise

1. Ex.5.54, p.197, 2

Exercise

1. Ex.5.54, p.197, 2
2. Ex.5.53, p.197, 4

Conditional introduction

1. We introduced conditional *elimination*
2. Next is conditional *introduction*

First, we need to talk about assumptions and subproofs.

Assumptions

In everyday arguments, we often make *assumptions*.

1. Assume I meet the love of my life tomorrow.
2. Assume I win a billion dollars.
3. Let's assume (for a second) that my client is guilty.

Reasoning with assumptions

After we make an assumption, we reason under that assumption.

1. Assume I meet the love of my life tomorrow. *It follows we'll get married!*
2. Assume I win a billion dollars. *It follows that I'll buy some fancy jewelry.*
3. Let's assume (for a second) that my client is guilty. *Then, you'd expect there to be forensic evidence linking him to the crime.*

The reasoning we do under the assumption is a **subproof**.

Assumption in proofs

When we assume in a proof, we

1. indent from the proof
2. write our assumption (justification is A)
3. draw a descending line to the left (until we are done with the assumption)

Here is how it looks:

1 Main Line

2		Assumption	A
3		.	
4		.	
5		.	

Assumptions in proofs

Suppose this is our proof setup:

1 $S \quad P$

Assumptions in proofs

Suppose this is our proof setup:

1 S P

We can make an assumption!

1 S P

2 $\mid B$ A
 $\underline{\hspace{1cm}}$

Assumptions in proofs

Suppose this is our proof setup:

1 S P

We can now reason "under" this assumption.

1	S	P
2	B	A
3	$S \wedge B$	$\wedge I$ 1, 2
4	.	
5	.	
6	.	

Assumptions in proofs

You can assume within a subproof:

1	Q	P
2	S	A
3	W	A
4	.	
5	.	
6	.	

Conditional introduction

1. We know how to assume
2. We can use rules to reason under the assumption
3. Certain rules allow you to reason to a wff from a subproof or subproofs.

Conditional introduction ($\rightarrow I$) is one of those rules.

Conditional introduction

Definition (Conditional Introduction ($\rightarrow I$))

If ψ is derived from the assumption ϕ , then $\phi \rightarrow \psi$ can be derived from the subproof started by ϕ .

Here is how it looks:

$$\begin{array}{lcl} n & | \phi & A \\ & | \vdots & \\ m & | \psi & \\ m+1 & \phi \rightarrow \psi & \rightarrow I \ n-m \end{array}$$

Conditional introduction (intuitively)

1 | Assume I get married.

| $\vdots$

7 | We have a baby.

8 | If I get married, then we'll have a baby.

A

some rule?

$\rightarrow I, 1-7$

Conditional introduction. Full Example

$$R \vdash Z \rightarrow R.$$

Conditional introduction. Full Example

$R \vdash Z \rightarrow R.$

First, set up the proof.

1 $R \quad P, Z \rightarrow R$

Conditional introduction. Full Example

$R \vdash Z \rightarrow R.$

Next, we make an assumption. Since we want $Z \rightarrow R$, assume Z

1	R	$P, Z \rightarrow R$
2	$\underline{Z}$	A, R

Now we want to derive R in the subproof.

Conditional introduction. Full Example

$R \vdash Z \rightarrow R.$

1	R	$P, Z \rightarrow R$
2	Z	A, R
3	$Z \wedge R$	$\wedge I\ 1, 2$
4	R	$\wedge E\ 3$

We have R ! Now let's use $\rightarrow I$.

Conditional introduction. Full Example

$R \vdash Z \rightarrow R.$

1	R	$P, Z \rightarrow R$
2	Z	A, R
3	$Z \wedge R$	$\wedge I\ 1, 2$
4	R	$\wedge E\ 3$
5	$Z \rightarrow R$	$\rightarrow I\ 2-4$

1. Our assumption is Z (line 2)
2. We derived R (line 4)
3. We used $\rightarrow I$ to put these together: $Z \rightarrow R$
4. Cite all of the lines of the subproof: 2-4

General tips about conditional introduction

1. You use $\rightarrow$ / a lot to derive conditionals.
2. If you want a conditional: $\phi \rightarrow \psi$, assume ϕ (leftside) and try to derive ψ (rightside)
3. Practice lots of proofs with $\rightarrow$ /

Exercise

1. Ex.5.55, p.206, 2

Exercise

1. Ex.5.55, p.206, 2
2. Ex.5.53, p.206, 4

Exercise

1. Ex.5.55, p.206, 2
2. Ex.5.53, p.206, 4
3. Ex.5.53, p.206, 7

Reiteration

Definition (Reiteration (R))

$\phi \vdash \phi$. From ϕ , the wff ϕ can be derived.

Example:

- | | | |
|---|-----|-----|
| 1 | P | P |
| 2 | P | R 1 |

Reiteration and simplification

Why R ? Look at this proof:

1	B	P
2	W	A
3	$W \wedge B$	$\wedge I, 1, 2$
4	B	$\wedge E, 3$
5	$W \rightarrow B$	$\rightarrow I 2-4$

If we could just use R , we could simplify this proof.

Reiteration and simplification

Why R ? Look at this proof:

1	B	P
2	W	A
3	B	$R, 1$
4	$W \rightarrow B$	$\rightarrow I\ 1-2$

Proof is simpler.

Reiteration: Clarifications

1. R let's you rewrite a wff on the same level of proof or **deeper into the proof**.
2. You cannot use R to "pull" wffs from inside a subproof to outside of the subproof.

1	R	P
2	$\begin{array}{ l} Z \end{array}$	A
3	Z	$R, 1.$ No! Wrong!

Exercise

We already did Ex.5.53, p.206, 7, but let's try it again now that we have R .

Negation Introduction and Elimination

Negation Introduction ($\neg I$) and Negation Elimination ($\neg E$) are sometimes called by other names:

1. Proof by contradiction
2. Reductio ad absurdum

Intuitively

- | | | |
|---|--------------------------------------|---------------|
| 1 | Assume God exists. | A |
| 2 | There is no evil. | |
| 3 | There is evil. Contradiction! | |
| 4 | God does not exist! | 1-3, $\neg$ / |

Negation Introduction and Elimination

Definition (Negation Introduction ($\neg I$))

From a subproof that assumes ϕ , derives ψ and $\neg(\psi)$, a wff $\neg(\phi)$ can be derived from the subproof.

Definition (Negation Elimination ($\neg E$))

From a subproof that assumes $\neg(\phi)$, derives ψ and $\neg(\psi)$, a wff ϕ can be derived from the subproof.

How to use the rules

How to use $\neg E$:

1. Assume $\neg(\phi)$
2. Show it leads to a **contradiction**: ψ and $\neg(\psi)$
3. Derive: ϕ

How to use $\neg I$:

1. Assume ϕ
2. Show it leads to a **contradiction**: ψ and $\neg(\psi)$
3. Derive: $\neg(\phi)$

k	ϕ	A
$\vdots$	$\vdots$	
n	ψ	
$n+1$	$\neg(\psi)$	

$n+2$

$\neg(\phi)$

$\neg I\ k-(n+1)$

k	$\neg(\phi)$	A
$\vdots$	$\vdots$	
n	ψ	
$n+1$	$\neg(\psi)$	

$n+2$

ϕ

$\neg E\ k-(n+1)$

◀ ◻ ▶
◀ ≡ ▶
◀ ≡ ▶
≡
↺ 🔍 ↻

Example

$$P \wedge \neg P \vdash S$$

Example

$$P \wedge \neg P \vdash S$$

$$1 \quad P \wedge \neg P \quad \text{P}$$

Example

$$P \wedge \neg P \vdash S$$

$$1 \quad P \wedge \neg P \quad P$$

$$2 \quad \underline{\neg S} \quad A$$

Example

$P \wedge \neg P \vdash S$

1	$P \wedge \neg P$	P
2	$\neg S$	A
3	P	$\wedge E$ 1
4	$\neg P$	$\wedge E$ 1

Example

$P \wedge \neg P \vdash S$

1	$P \wedge \neg P$	P
2	$\neg S$	A
3	P	$\wedge E$ 1
4	$\neg P$	$\wedge E$ 1
5	S	$\neg E$ 2-4

Clarifications

1. With these rules, you assume the **opposite** of the conclusion you want to derive.
2. Useful if you've exhausted other options

Exercise

1. 5.57, 2
2. 5.57, 3

Disjunction Introduction

Definition (Disjunction Introduction ($\vee I$))

$\phi \vdash \phi \vee \psi$ or $\phi \vdash \psi \vee \phi$

Disjunction Introduction: Intuitively

- ▶ P1: I will go to the store.
- ▶ C: Therefore, I will go to the store or the movies.

For any proposition P , we can derive $P \vee \varphi$.

Disjunction Introduction: Example 1

1	P	P
2	$P \vee Q$	$\vee I\ 1$

Disjunction Introduction: Example 2

1	A	P
2	$(A \vee Q) \rightarrow R$	P, R
3	$A \vee Q$	$\vee I$ 1
4	R	$\rightarrow E$, 2,3

Disjunction Elimination

Definition (Disjunction Elimination ($\vee E$))

From $\phi \vee \psi$ and two derivations of χ —one involving ϕ as an assumption in a subproof, the other involving ψ as an assumption in a subproof—we can derive χ from those subproofs.

n	$\phi \vee \psi$	P
k	ϕ	A
	$\vdots$	
k+i	χ	
j	ψ	A
	$\vdots$	
j+1	χ	
m	χ	$\vee E$ n, k-(k+i), j-(j+1)

Disjunction Elimination: Intuitively

1 I will stay home or party P

Disjunction Elimination: Intuitively

1	I will stay home or party	P
2	Assume I stay home	A
3	$\vdots$	

Disjunction Elimination: Intuitively

1	I will stay home or party	P
2	Assume I stay home	A
3	⋮	
4	Assume I party	A
5	⋮	

Disjunction Elimination: Intuitively

1	I will stay home or party	P
2	<u>Assume I stay home</u>	A
3	⋮	
4	I have a great time	
5	<u>Assume I party</u>	A
6	⋮	
7	I have a great time	

Disjunction Elimination: Intuitively

1	I will stay home or party	P
2	<u>Assume I stay home</u>	A
3	$\vdots$	
4	I have a great time	
5	<u>Assume I party</u>	A
6	$\vdots$	
7	I have a great time	
8	I have a great time	1-7, $\vee E$

Disjunction Elimination: Example

1	$P \vee Q$	P
2	$P \rightarrow R$	P
3	$Q \rightarrow R$	P
4	$\begin{array}{ l} P \\ \hline \end{array}$	A
5	$\begin{array}{ l} R \\ \hline \end{array}$	$\rightarrow E\ 2,4$
6	$\begin{array}{ l} Q \\ \hline \end{array}$	A
7	$\begin{array}{ l} R \\ \hline \end{array}$	$\rightarrow E\ 3,6$
8	R	$\vee E\ 1, 4-5, 6-7$

Disjunction Elimination: Clarifications

1. Similar versions: proof by cases, constructive dilemma
2. Consider using when you have a disjunction
3. People struggle with this type of reasoning, e.g., lawyers, professionals, etc.
4. Very useful in daily life.
 - ▶ You have two options A and B
 - ▶ You don't know which is true (or which to choose)
 - ▶ If C follows from both, then C (or pick C)

Exercise

1. 5.58, 2
2. 5.58, 5

Biconditional Elimination

Definition (Biconditional Elimination ($\leftrightarrow E$))

$\phi \leftrightarrow \psi, \phi \vdash \psi$ or $\phi \leftrightarrow \psi, \psi \vdash \phi$

Biconditional Elimination: Example 1

1	$P \leftrightarrow \neg Q$	P
2	$\neg Q$	P
3	P	$\leftrightarrow E$ 1, 2

Biconditional Elimination: Key Ideas

1. You need a biconditional $\phi \leftrightarrow \psi$ and one of the sides to reason to the other side.
2. Don't confuse it with $\rightarrow E$: $\phi \rightarrow \psi, \phi \vdash \psi$

Biconditional Introduction

Definition (Biconditional Introduction ($\leftrightarrow I$))

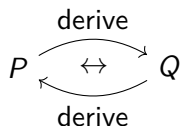
From two subproofs: (1) ϕ is assumed and ψ is derived; (2) ψ is assumed and ϕ is derived, a wff $\phi \leftrightarrow \psi$ can be derived.

n	ϕ	A
	$\vdots$	
n+i	ψ	
k	ψ	A
	$\vdots$	
k+j	ϕ	
m	$\phi \leftrightarrow \psi$	$\leftrightarrow I$ n+1, k+j

Biconditional Introduction: Visualized

Biconditional introduction involves two steps (two subproofs):

1. Assume P , derive Q .
2. Assume Q , derive P .



Biconditional Introduction: Example

1	P	P
2	Q	$P, P \leftrightarrow Q$
3	P	A
4	Q	$R\ 2$
5	Q	A
6	P	$R\ 1$
7	$P \leftrightarrow Q$	$\leftrightarrow\ /\ 3-4, 5-6$

Biconditional Rules: Exercise

- ▶ Ex. 5.59, 1
- ▶ Ex. 5.59, 4

Strategies: Introduction

You now have all the rules you need to solve everything. **BUT ...**

1. Solving proofs is still difficult
2. Hard to know where to start after setting it up

Strategy 1: Slam E rules

- ▶ Some proofs have premises.
- ▶ Elimination rules give you simpler wffs
- ▶ So, use as many **elimination** rules as you can

Strategy 1: Example

This proof has a lot of premises. Let's use a bunch of elimination rules:

- 1 $P \wedge Q$ P
- 2 $Q \rightarrow R$ P
- 3 $R \leftrightarrow S$ P
- 4 S P, S

Strategy 1: Example

1	$P \wedge Q$	P
2	$Q \rightarrow R$	P
3	$R \leftrightarrow S$	P
4	S	P, S
5	$\boxed{P}$	$\wedge E$ 1
6	$\boxed{Q}$	$\wedge E$ 1

Strategy 1: Example

1	$P \wedge Q$	P
2	$Q \rightarrow R$	P
3	$R \leftrightarrow S$	P
4	S	P, S
5	P	$\wedge E$ 1
6	Q	$\wedge E$ 1
7	$\boxed{R}$	$\rightarrow E$ 2,6

Strategy 1: Example

1	$P \wedge Q$	P
2	$Q \rightarrow R$	P
3	$R \leftrightarrow S$	P
4	S	P, S
5	P	$\wedge E$ 1
6	Q	$\wedge E$ 1
7	R	$\rightarrow E$ 2,6
8	$\boxed{S}$	$\leftrightarrow E$ 3,7

We solved the proof by repeatedly using Elim-rules.

Strategy 2: Use Intro-rules for what you want

- ▶ Sometimes you want a specific wff (e.g., the conclusion)
- ▶ Every wff will be a wff of a specific type, e.g., conditional, conjunction
- ▶ Use an **introduction** rule for that wff type.

Strategy 2: Example

$$1 \quad Q \wedge P \quad P, (S \wedge \neg W) \rightarrow Q$$

Oh no! Look at that conclusion! It contains letters we don't even have in the premises!

Strategy 2: Example

$$1 \quad Q \wedge P \quad P, (S \wedge \neg W) \rightarrow Q$$

Strategy 2: We want a conditional, so let's use $\rightarrow I$.

Strategy 2: Example

1	$Q \wedge P$	$P, (S \wedge \neg W) \rightarrow Q$
2	$\begin{array}{ l} S \wedge \neg W \\ \hline \end{array}$	A, Q
3	Q	$1 \wedge E$
4	$(S \wedge \neg W) \rightarrow Q$	$\rightarrow I, 2-3$

Exercise

1. 5.60, 7

Exercise

1. 5.60, 7
2. Typo in Ex 5.61, 6. Should be: $\neg R \vdash (P \vee \neg W) \rightarrow (Q \vee \neg R)$

Disjunctive Syllogism

Definition (Disjunctive Syllogism (DS))

$\phi \vee \psi, \neg(\psi) \vdash \phi$ or $\phi \vee \psi, \neg(\phi) \vdash \psi$

DS. Simple Example

- | | | |
|---|-----------------------|---------|
| 1 | $P \vee (R \wedge S)$ | P |
| 2 | $\neg(R \wedge S)$ | P |
| 3 | P | DS 1, 2 |

Modus Tollens

Definition (Modus Tollens (MT))

$$\phi \rightarrow \psi, \neg(\psi) \vdash \neg(\phi)$$

MT. Simple Example

1	$S \rightarrow D$	P
2	$\neg D$	P
3	$\neg S$	1,2 MT

Hypothetical Syllogism

Definition (Hypothetical Syllogism (HS))

$$\phi \rightarrow \psi, \psi \rightarrow \chi \vdash \phi \rightarrow \chi$$

HS. Simple Example.

- 1 $P \rightarrow R$ P
- 2 $Q \rightarrow R$ P
- 3 $P \rightarrow R$ HS 1, 2

Derived Rules: Exercise

- ▶ Ex. 5.62, 1
- ▶ Ex. 5.62, 2

Double negation

Definition (Double Negation (DN))

$$\neg\neg(\phi) \dashv\vdash \phi$$

DN. Simple Example

1	$P \rightarrow R$	$P, \neg\neg(P \rightarrow R)$
2	$\neg\neg(P \rightarrow R)$	DN 1

DN. Subformula

1 $P \vee (R \wedge S)$ P, $(P \vee (R \wedge S))$

2 $(P \vee (R \wedge \neg\neg S))$ DN 1

Notice that DN can be applied to subformula rather than the entire wff.

DN. Incorrect Usage.

1	$P \vee (R \wedge S)$	P, $(P \vee (R \wedge S))$
2	$\neg P \vee (\neg R \wedge S)$, NO!	DN 1

De Morgan's Laws

Definition (De Morgan's Laws (DeM))

$$\neg(\phi \vee \psi) \dashv\vdash \neg(\phi) \wedge \neg(\psi)$$

$$\neg(\phi \wedge \psi) \dashv\vdash \neg(\phi) \vee \neg(\psi)$$

DEM

The diagram illustrates the De Morgan's Law (DEM) for disjunction. It shows the expression $\neg(\phi \vee \psi)$ on the left. Two curved arrows originate from the disjunction symbol $\vee$ and point to the variables ϕ and ψ respectively. A single curved arrow originates from the negation symbol $\neg$ and points to the expression $\neg\phi \wedge \neg\psi$ on the right, indicating the transformation of the expression.

$$\neg(\phi \vee \psi) \rightarrow \neg\phi \wedge \neg\psi$$

DEM. Simple Example

- 1 $\neg(P \vee Q)$ P
- 2 $\neg P \wedge \neg Q$ DeM 1
- 3 $\neg(P \vee Q)$ DeM 2

DEM. Simple Example

$$1 \quad \neg(A \vee B) \quad P, \neg B$$

DEM. Simple Example

1 $\neg(A \vee B)$ P, $\neg B$

2 $\neg A \wedge \neg B$ DeM 1

DEM. Simple Example

- | | | |
|---|------------------------|--------------|
| 1 | $\neg(A \vee B)$ | P, $\neg B$ |
| 2 | $\neg A \wedge \neg B$ | DeM 1 |
| 3 | $\neg B$ | $\wedge E$ 2 |

Implication

Definition (Implication (IMP))

$$\phi \rightarrow \psi \dashv\vdash \neg(\phi) \vee \psi$$

IMP. Simple example.

1	$P \rightarrow Q$	P
2	$\neg P \vee Q$	IMP 1
3	$P \rightarrow Q$	IMP 2

IMP. Negated conditionals.

$$1 \quad \neg(P \rightarrow Q) \quad P, \neg Q$$

IMP. Negated conditionals.

$$1 \quad \neg(P \rightarrow Q) \quad P, \neg Q$$

$$2 \quad \neg(\neg P \vee Q) \quad 1 \text{ IMP}$$

Notice that IMP is applied to $P \rightarrow Q$

IMP. Negated conditionals.

1	$\neg(P \rightarrow Q)$	$P, \neg Q$
2	$\neg(\neg P \vee Q)$	1 <i>IMP</i>
3	$\neg\neg P \wedge \neg Q$	2 <i>DeM</i>
4	$\neg Q$	3 <i>$\wedge E$</i>

Equivalence Rules: Exercise

- ▶ Ex. 5.63, 1
- ▶ Ex. 5.63, 2
- ▶ Ex. 5.63, 3

Tips

To get ready for Exam 3, you should do the following:

1. Do all of the the homework (don't wait!)
2. Do the practice exam
3. If some proof is difficult for you, take a picture and email me it.
4. Attend the optional classes!