Introduction to Symbolic Logic

David W. Agler

RL: Beyond Predicate Logic

Predicate Logic Semantics with Variable Assignments

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Predicate Logic using Names

Recall the following valuation rules for predicate logic (let $\alpha_1, \ldots, \alpha_n$ be any series of names (not necessarily distinct), P be any n-place predicate, and ϕ, ψ are wffs in RL):

Definition (RL-valuation using names)

1. if $P\alpha_1 \ldots \alpha_n$ is a closed atomic wff in RL, then $v_{\mathcal{M}}(P\alpha_1 \ldots \alpha_n) = T$ iff $\langle \mathscr{I}(\alpha_1), \ldots, \mathscr{I}(\alpha_n) \rangle \in \mathscr{I}(P)$, otherwise $v_{\mathcal{M}}(P\alpha_1 \ldots \alpha_n) = F$
2. $v_{\mathcal{M}}(\neg(\phi)) = T$ iff $v_{\mathcal{M}}(\phi) = F$
3. $v_{\mathcal{M}}(\phi \wedge \psi) = T$ iff $v_{\mathcal{M}}(\phi) = T$ and $v_{\mathcal{M}}(\psi) = T$
4. $v_{\mathcal{M}}(\phi \vee \psi) = T$ iff $v_{\mathcal{M}}(\phi) = T$ or $v_{\mathcal{M}}(\psi) = T$
5. $v_{\mathcal{M}}(\phi \rightarrow \psi) = T$ iff $v_{\mathcal{M}}(\phi) = F$ or $v_{\mathcal{M}}(\psi) = T$
6. $v_{\mathcal{M}}(\forall x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x)$ for every name α in RL.
7. $v_{\mathcal{M}}(\exists x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x)$ for at least one name α in RL.

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Two Problems: Problem 1

This definition of the valuation function has at least two problems:

Problem 1: the valuation function is only defined for **closed RL-wffs**. It is **undefined** for open RL-wffs (wffs with free variables, e.g. Px). A more inclusive valuation function might be desirable for a few reasons:

- **special wffs:** Ixx where I is the two-place identity predicate
- **compositional concerns:** shouldn't the truth value of $(\exists x)Px$ be determined by the existential quantifier and Px, rather than say a wff $Pa \vee Pb, \ldots Pn$?
- **tighter relation between syntax and semantics:** in some systems open formulas are wffs, but they are not interpreted. A valuation function that covers open formulas would mean that all wffs are interpreted

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Problem 2: it assumes that there is an RL-name for every item in the domain.

- Notice: we specify the value of existentially and universally quantified wffs in terms of non-quantified wffs. Example: $v_{\mathcal{M}}(\forall x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x)$ for every name α in RL.
- the issue isn't that we don't have enough names
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Some objects may not be named

Cats that are pets are usually named; but some stray cats may be unnamed.

The key idea

- The key idea behind fixing both problems is to treat variables like **pronouns** rather than **names**.
- In "It is happy" (Hx), the pronoun "it" can refer potentially to any item in the domain. Its truth or falsity will depend upon the referent of x or "it"
- since the referent of pronouns can vary and they can potentially refer to any item in the domain, we can use this feature to generalize about objects in the domain

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Fixing the first problem: The key idea

With respect to the first problem:

- We can assign a truth value to Px "he (or she or it) is a person" if there is a way of identifying the referent of the pronoun "he".
- The truth value of such a wff will depend upon the referent of "he".
- If "he" designates something that is not a person, then Px is false; while if it identifies something that is a person, then Px is true.

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Fixing the second problem: The key idea

With respect to the second problem:

- $(\exists x)Px$ "someone is a person" is true iff there is at least one way of identifying the referent of "he" such that the object is a person.
- In short: $v(\exists x)P = T$ iff $v(Px) = T$ for at least one referent of pronoun x
- $(\forall x)Px$ "everyone is a person" is true iff on every way of identifying the referent of "he" that object is a person.
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Variable assignment

- In order to fix both problems, we will introduce some additional technical apparatus.
- Let's begin with the notion of a **variable assignment**.

Definition (variable assignment)

A variable assignment g for a model $\mathcal{M}$ ($\langle \mathcal{D}, \mathscr{I} \rangle$) is a function that assigns to each variable α some object in $\mathcal{D}$.

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Variable assignment: notation

- We need a way to specify variable assignments so that it is clear which item in the domain is assigned to which variable in the language
- We will use g to stand for a variable assignment
- "$g(x)$" will specify the variable assignment of x
- "$g(x)$" reads the variable assignment g that takes x as input (it will yield an item from the domain as a value).

Example

1. $g(x) = u_1$ assigns u_1 from the domain to the variable x
2. $g(y) = u_2$ assigns u_2 from the domain to the variable y
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Relativizing the valuation function

- The next step is to relativize the **valuation function** not merely to a model ($\mathcal{M}$) but also to a variable assignment (g)
- Not simply $v_{\mathcal{M}}(\phi)$ but $v_{\mathcal{M},g}(\phi)$
- Under this valuation function, wffs are true or false with respect to a **model** ($\mathcal{M}$) and a variable assignment (g)
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Valuation function for closed and open atomic wffs

This relativization allows us to formulate two different rules for atomic wffs in RL (let α be any name and x be any variable):

Definition

1a if $P\alpha_1 \dots \alpha_n$ is a closed atomic wff in RL, then $v_{\mathcal{M},g}(P\alpha_1 \dots \alpha_n) = T$ iff $\langle \mathscr{I}(\alpha_1), \dots, \mathscr{I}(\alpha_n)\rangle \in \mathscr{I}(P)$. Otherwise, $v_{\mathcal{M},g}(P\alpha_1 \dots \alpha_n) = F$

1b if $Px_1 \dots x_n$ is an open atomic wff in RL, then $v_{\mathcal{M},g}(Px_1 \dots x_n) = T$ iff $\langle g(x_1), \dots, g(x_n)\rangle \in \mathscr{I}(P)$. Otherwise, $v_{\mathcal{M},g}(Px_1 \dots x_n) = F$

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1. does not change how we evaluate closed atomic wffs
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- Notice that we now can define the truth value of wffs that have free variables.
- Take the wff *Ixx* where *I* is the two-place predicate "x is identical to x".
- $v_{\mathcal{M},g}(Ixx) = T$ iff $\langle g(x), g(x) \rangle \in \mathscr{I}(I)$.
- In other words, "x is identical to x" is true (relative to the model and the variable assignment) if and only if the ordered pair consisting of the variable assignment $g(x)$ and $g(x)$ is in the interpretation of I
- Put even more plainly: if the objects picked out by $g(x)$ are identical to each other, then "x is identical to x" is true

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Open wffs can be assigned truth values

$v(Cx) = T$ ("x is a cat") is true iff $g(x)$ assigns x to an item in the interpretation of C. That is, iff $g(x) \in \mathscr{I}(C)$.

There is still a problem!

Problem!

- valuation rules apply only to atomic wffs containing either names or variables but not to wffs containing **both names and variables**
- The valuation rule works for Pa, Lab, Px, Lxx
- BUT NOT for Lax, Lxa (names and variables)
- the valuation rule is **undefined** for these wffs

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Generalizing the valuation function

To solve this problem, we will need to do two things:

1. define the notion of a **term** that includes names and variables
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RL-Term (name or variable)

First, let's define the notion of an **RL-term**:

Definition (RL-term)

An RL-term t is any name or variable in RL.

Example

1. x is a variable; therefore it is a term
2. y is a variable; therefore it is a term
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Denotation of an RL-term

Second, let's define and introduce some notation for the denotation of a term.

- Let the expression $[t]_{\mathcal{M},g}$ read the denotation of the term t relative to a model $\mathcal{M}$ and variable assignment g.

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Let $\mathcal{M}$ be a model, g be a variable assignment, t be a term (name or variable). The denotation of t relative to a model and a variable assignment (that is, $[t]_{\mathcal{M},g}$) is:

1. $\mathscr{I}(t)$ if t is an RL-name, or
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Let's look at some examples of the denotation of a term. To do this, we will need part of a model and a variable assignment. So first consider the following:

- $\mathcal{D} : \{1, 2, 3\}$
- $\mathscr{I}(a) = 1, \mathscr{I}(b) = 2, \mathscr{I}(c) = 3$
- $g(x) = 1, g(y) = 2, g(z) = 1$

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Generalized valuation function for atomic wffs

We can now combine the two valuation functions into a single valuation rule that makes us of the notion of a denotation of a term.

Definition

if t is a term, P is an n-place predicate, and $Pt_1 \ldots t_n$ is an atomic wff in RL, then $v_{\mathcal{M},g}(Pt_1 \ldots t_n) = T$ iff $\langle [t_1]_{\mathcal{M},g}, \ldots, [t_n]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$

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Examples

- take *Lax*, an atomic wff containing the name *a* and variable *x* ("Al loves *x*".)
- $v_{\mathcal{M},g}(Lax) = T$ iff $\langle [a]_{\mathcal{M},g}, [x]_{\mathcal{M},g} \rangle \in \mathscr{I}(L)$
- $v_{\mathcal{M},g}(Lax) = T$ iff the ordered pair consisting of the denotation of the name *a* and the denotation of the variable *x* are in the interpretation of *L*
- "Al loves x" is true provided, relative to a model and relative to a variable assignment, the ordered pair $\langle Al, [x] \rangle$ is in the interpretation of the two-place predicate *Lxy* (*x* loves *y*).

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Problem 2: Quantified wffs and names

- We have a solution for Problem 1.
- But Problem 2 remains. That is, we are still unpacking the truth value of quantified wffs using names
- One initial thought is to say $v_{\mathcal{M},g}(\exists x)Px = T$ iff $v_{\mathcal{M},g}P(\alpha/x) = T$ (for some name α) or $v_{\mathcal{M},g}(Px) = T$.
- Promising approach since we have a procedure for determining the truth value of Px relative to g; namely, $v_{\mathcal{M},g}(Px) = T$ iff $\langle [x]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$. Thus, $v_{\mathcal{M},g}(\exists x)Px = T$ iff $\langle [x]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$
- Also an attractive option since the truth value of $(\exists x)Px$ is determined by its parts: the existential quantifier and Px.

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- Does not get us the right result since the variable assignment g takes each variable and assigns it a single item from the domain.
- This means that $g(x)$ refers to a **single item** in the domain
- Problematic because an existential quantified wff Px is true not so long as **the single item** picked out by the denotation of x is in P, but so long as *at least one* item from the domain is in the interpretation of P.

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Something is a cat

Suppose $\mathcal{D} : \{Jon, Snickers\}$ where Jon is a person and Snickers is a cat. Notice that $g(x) = Jon$ and that $[x]_{\mathcal{M},g} \notin \mathscr{I}(C)$; therefore, $v(Cx) = F$; therefore, $v(\exists x)Cx = F$.

Problem 2: Quantified wffs and names

In other words:

- we cannot specify the truth value of quantified wffs using variable assignments alone
- we need a way of specifying the truth value of a wff like $(\exists x)Px$ such that this wff is true if there is at least one variable assignment $g(x)$ such that $g(x) \in \mathscr{I}(P)$
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Variant variable assignments

- Let's introduce the notion of a **variant variable assignment**:

Definition (variant variable assignment)

Let α be a variable and u be an item in the domain $u \in \mathcal{D}$ of a model, a variant variable assignment g_u^α is a variable assignment g for a model $\mathcal{M}$ except that it assigns u to α.

Reading variant variable assignment notation

1. g_u^α is read as the variable assignment g except that the variable α is assigned the item u from the domain
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Example 1 of Variant Variable assignment

Example (Illustration of a variant variable assignment)

Suppose there is a variable assignment g where $g(x) = u_1, g(y) = u_2, g(z) = u_3$. Now let's consider one variant variable assignment: $g^y_{u_1}$.

- $g : g(x) = u_1, g(y) = u_2, g(z) = u_3$
- $g^y_{u_1} : g^y_{u_1}(x) = u_1, g^y_{u_1}(y) = u_1, g^y_{u_1}(z) = u_3$

Notice that the only difference between g and $g^y_{u_1}$ is that $g^y_{u_1}$ assigns the variable y to u_1 instead of u_2.

Example 2 of Variant Variable assignment

A **variable assignment** and a **variant variable assignment** might be identical.

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Variant Variable Assignments

Question

How can variant variable assignments be used to define a new valuation function that will deal with the problem involving quantified wffs?

- $v(\exists x)_{\mathcal{M},g} Px = T$ iff
  - there is at least one item $u \in \mathcal{D}$ such that $v_{\mathcal{M}g^x_u}(Px) = T$
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Definition of a valuation function using variant variable assignments

An **RL-valuation** — for a model $\mathcal{M}$ and variable assignment g — is a function that assigns to each RL-wff a truth value (T or F) using the following rules (let P be any n-place predicate, $t_1, \ldots, t_n$ be a series of terms (not necessarily distinct), α be any variable, ϕ, ψ any RL-wff):

1. $v_{\mathcal{M},g}(Pt_1 \ldots t_n) = T$ iff $\langle [t_1]_{\mathcal{M},g}, \ldots, [t_n]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$
2. $v_{\mathcal{M},g}(\neg(\phi)) = T$ iff $v_{\mathcal{M},g}(\phi) = F$
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5. $v_{\mathcal{M},g}(\phi \rightarrow \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = F$ or $v_{\mathcal{M},g}(\psi) = T$
6. $v_{\mathcal{M},g}(\forall\alpha)\phi = T$ iff for every $u \in \mathcal{D}$, $v_{\mathcal{M},g^{\alpha}_{u}}(\phi) = T$.
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Solution to Problem 2

- Recall that the problem with unpacking the truth value of $(\exists x)Px$ in terms of Pa and Pb was that there was no guarantee that every item in the domain was named
- Recall also that the problem with unpacking the truth value of $(\exists x)Px$ in terms of Px relative to a variable assignment g was that there are cases where $v(\exists x)Px = T$ but $v_g(Px) = F$
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But wait!

What about our cat example? What about Snickers?

Something is a cat

Suppose $\mathcal{D}: \{Jon, Snickers\}$, $\mathscr{I}(C) = \{Snickers\}, g(x) = Jon$. Notice $v(\exists x)_{\mathcal{M},g} Cx = T$ since $v_{\mathcal{M},g^x_{Snickers}} Cx = T$ since there is a cat but $[x]_{\mathcal{M},g^x_{Snickers}} \in \mathscr{I}(C)$

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We saw that using names to unpack the valuation function had two potential problems:

Problem 1: it left open wffs undefined. We solved this by relativizing the valuation function to **variable assignments**

- Allows us to specify the truth value of wffs like Ixx
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Problem 2: it assumes that there is an RL-name for every item in the domain. We solved this by relativizing the valuation function to **variable assignments** *and* making the truth of quantified wffs depend upon variant variable assignments

- Not a problem if every item in the domain isn't named
- variables can ensure that each item is referenced in some way
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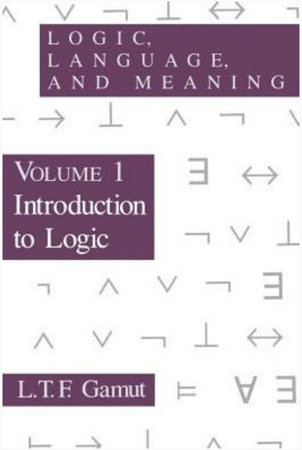
Resources

- Gamut, L.T.F. 1991. *Language, Logic, and Meaning: Volume I Introduction to Logic.* Chicago: The University of Chicago Press.



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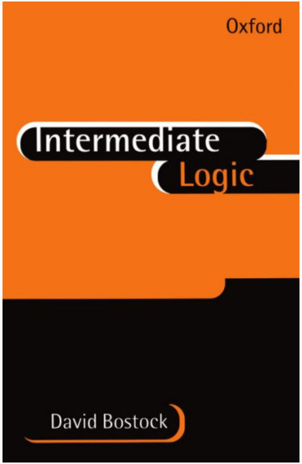
- Bostock, David. 1997. *Intermediate Logic.* Oxford: Oxford University Press.



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- Sider, Theodore. 2010. *Logic for Philosophy.* Oxford: Oxford University Press.