Handout 3

Truth Tables

In Chapter 1, a method was devised for determining whether or not arguments are valid or invalid. This test required the use of our psychological powers. There, it was said that an argument can be identified as *deductively valid* if and only if it was impossible to imagine a scenario where all of the premises were true and conclusion false. If it is possible to imagine such a scenario, then the argument is invalid.

It was also noted there that the use of the "imagination test" faced a number of problems in its application to arguments expressed in English. Some arguments consist of many premises and the state of affairs that these premises express can be difficult to picture in one's mind. In addition, it was noted that human beings are subject to certain biases and so they might be tempted to say that certain arguments are "invalid" when their conclusions conflict with their beliefs or "valid" when their conclusions accord with their desires. Finally, since arguments are expressed in a language that allows for ambiguity, the contents of an argument may be imagined differently by different individuals.

What we would like then is a test that does not rely upon the limited imaginative powers of human beings, whose use is independent of human bias, and that is expressed using a language that does not permit ambiguity.

In this handout, an alternative method for determining the validity of an argument is proposed. This is the method of truth tables.

> DEFINITION – TRUTH TABLE
>
> A truth table for **PL** is a table that provides a graphical way of representing valuations of wff(s) under a set of interpretations.

A truth table can be used to mechanically test sets of wffs and arguments for different properties. Mechanical tests that can test, in a finite number of steps, whether an argument is deductively valid or whether a set of propositions can all be true is known as a “decision procedure”.

DEFINITION – DECISION PROCEDURE

A decision procedure is a mechanical method that determines in a finite number of steps whether a proposition, set of propositions, or argument has a certain logical property (one of these being whether or not an argument is deductively valid!).

3.1 Using Truth Tables to Determine the Truth Value of a Complex Wff

Step 1: Write down the wff.
Step 2: Write the truth value (T or F) under each propositional letter in the wff for each interpretation $\mathscr{I}$.
Step 3: Starting with the truth-functional operator with the least scope and proceeding to the truth-functional operator with the most scope, use the appropriate valuation (truht-table) rule to determine the truth value of the complex proposition.

P	$\neg P$
T	F
F	T

Truth Table 3.1 – Truth Table: Negation

P	R	$P \wedge R$	$P \vee R$	$P \to R$	$P \leftrightarrow R$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Truth Table 3.2 – Truth Table: Conjunction, Disjunction, Conditional, and Biconditional

Rather than write $\mathscr{I}(P) = T$, we write $v(P) = T$

To start, consider a simple example of how to determine $P \to \neg R$ under a single interpretation of P and R: $v(P) = T$ and $v(R) = F$. First, we write out the formula or set of formulas you want to test (see Truth Table 3.3).

P	$\to$	$\neg$	R

Truth Table 3.3 – Truth Value for $P \to \neg R$

Next, we write the truth values below propositional letters. Since $v(P) = T$, write T under P and since $v(R) = F$, write F under R (see Truth Table 3.4).

P	$\rightarrow$	$\neg$	R
T			F

Truth Table 3.4 – Truth Value for $P \rightarrow \neg R$

Next, starting from the truth-functional operator(s) with the least scope and using the valuation rule that corresponds to that operator, determine the truth value of the complex subformula that contains that operator (see Truth Table 3.5).

P	$\rightarrow$	$\neg$	R
T		T	F

Truth Table 3.5 – Truth Value for $P \rightarrow \neg R$

Finally, use the valuation rules and the truth values of complex subformula to determine the truth value of larger and larger subformula until you have determined the truth value of the entire formula. The truth value written under the main operator will be the truth value of the entire formula (see Truth Table 3.6).

P	$\rightarrow$	$\neg$	R
T	T	T	F

Truth Table 3.6 – Truth Value for $P \rightarrow \neg R$

Exercise 1: *Let $v(P) = T$, $v(R) = F$ and $v(S) = T$*

1. $\neg P \rightarrow R$
2. $\neg(P \vee R)$
3. $\neg(\neg P \leftrightarrow \neg S)$

3.2 The Truth-Table Method

In the above examples, the truth value of complex wffs are determined under one interpretation of the propositional letters. The truth-table method, however, is more general than this as it allows for determining the truth value of wffs under all admissible interpretations of the propositional letters. For example, consider the following wff: $\neg P \vee \neg R$.

Step #1: Write out all of the propositional letters in the formula in a separate column and the formula or formulas you want to test to the right of it:

P	R	$\neg$	P	$\vee$	$\neg$	R

Truth Table 3.7 – Truth Table for $\neg P \vee \neg R$

Step #2: consider the different possible interpretations for the propositional letters in a wff:

P	R	$\neg$	P	$\vee$	$\neg$	R
T	T					
T	F					
F	T					
F	F					

Truth Table 3.8 – Truth Table for $\neg P \vee \neg R$

Step #3: for each row, write the truth values under the corresponding letter in the row.

P	R	$\neg$	P	$\vee$	$\neg$	R
T	T		T			T
T	F		T			F
F	T		F			T
F	F		F			F

Truth Table 3.9 – Truth Table for $\neg P \vee \neg R$

Step #4: Start from the truth-functional operators with the least scope to the operators with the most scope (main operator), determine the truth value of formula upon which these operators operate upon, and work your way to determining the truth value of the main operator.

P	R	$\neg$	P	$\vee$	$\neg$	R
T	T	F	T		F	T
T	F	F	T		T	F
F	T	T	F		F	T
F	F	T	F		T	F

Truth Table 3.10 – Truth Table for $\neg P \vee \neg R$

P	R	$\neg$	P	$\vee$	$\neg$	R
T	T	F	T	F	F	T
T	F	F	T	T	T	F
F	T	T	F	T	F	T
F	F	T	F	T	T	F

Truth Table 3.11 – Truth Table for $\neg P \vee \neg R$

Truth Table 3.11 shows the truth value of $\neg P \vee \neg R$ under all of the different ways that P and R can be interpreted in **PL**.

Exercise 2: *Determine the truth values of the following wffs under all of the different ways that the propositional letters in that wff can be interpreted.*

1. $P \to \neg R$
2. $(P \land R) \to R$
3. $\neg\neg(P \leftrightarrow R) \lor Z$

3.3 Truth-Table Analysis

Thus far, we have shown how to use truth tables to determine the truth value of a complex wff under an interpretation. Truth tables have an additional use in that they can be employed to determine whether a certain wff or set of wffs has a certain property. This, it will be shown, is useful for it allows us a way to determine whether propositions, groups of propositions, or arguments in English have properties that interest us, e.g. whether an argument is deductively valid.

3.3.1 Truth Tables and Validity

In this section, the notions of "semantic consequence", "valdity" and "invalidity" are defined as well as how to analyze a truth table to determine if an argument is deductively valid or invalid.

With respect to **PL** we say that a wff **Q** is a "semantic consequence" of a set of wffs Γ if and only if there is no interpretation such that all of the wffs in Γ are true and Q is false.

Definition – semantic consequence (entailment)

A wff **Q** is a *semantic consequence* (entailment) in **PL** of a set of wffs Γ if and only if there is no interpretation $\mathscr{I}$ in which all of the members of Γ are true and **Q** is false.

The double turnstile '$\models$' ("models" or "entails") expresses the fact that there is no interpretation of the wffs to the left of the turnstile such that these wffs are true and the wff to the right of the turnstile is false.

The notion of "semantic consequence" concerns the relationship between *any* set of wffs Γ and *any* wff **Q**. An argument is a series of propositions where one proposition (the conclusion) is represented as following from another set of propositions (the premises). The expression "following from" is ambiguous. On the one hand, it might express the idea that we can "prove" the conclusion from the premises. This is the "proof-theoretic" or "syntactic" notion of validity. On the other hand, it might express the notion of semantic consequence, i.e., that there is no way to interpret the premises and the conclusion such that the premises are true and the conclusion is false. This is the "model-theoretic" or "semantic" notion of validity. Here we define what it means to be "valid" in the semantic way.

DEFINITION – VALID IN **PL**

An argument $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$ therefore $\mathbf{Z}$ in **PL** is deductively valid if and only if $\mathbf{Z}$ is a semantic consequence of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$. That is, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$ therefore $\mathbf{Z}$ is deductively valid if and only if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y} \models \mathbf{Z}$.

DEFINITION – INVALID IN **PL**

An argument $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$ therefore $\mathbf{Z}$ in **PL** is deductively invalid if and only if $\mathbf{Z}$ is **not** a semantic consequence of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$. That is, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y}$ therefore $\mathbf{Z}$ is deductively invalid if and only if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Y} \not\models \mathbf{Z}$.

DEFINITION – TRUTH-TABLE TEST FOR VALIDITY

A truth table shows that an argument is valid if and only if there is no row of the truth table where the premises are true and the conclusion is false. Equivalently, if there is no row of the truth table where the premises and the negation of the conclusion are true.

DEFINITION – TRUTH-TABLE TEST FOR INVALIDITY

A truth table shows that an argument is invalid if and only if there is a row of the truth table where the premises are true and the conclusion is false.

For example, consider the argument $P \to Q, \neg(P \lor Q)$ therefore P. We might simply write the premises and the conclusion as wffs in the truth table and then check whether there is any row where the premises are true and the conclusion is false. If there is, then the argument is invalid. If there isn't, then the argument is valid.

Alternatively, we might take the premises $P \to Q, \neg(P \lor Q)$ and the negation of P, i.e. $\neg P$, and check if there is any row where all of the wffs are true. If there are, then it is possible for the premises to be true and the conclusion false. And so, the argument would be invalid. If there is no row where all of the wffs are true, then the argument is valid.

Exercise 3: *Using the truth-table method, show the following semantic entailments:*

1. $P, P \to R \models R$
2. $\neg P, \neg P \lor R \models R$
3. $P \lor R, \neg R, P \to R \models R$
4. $J \to C, \neg C \models \neg J$
5. $J \leftrightarrow C, C \models J \lor \neg\neg C$

P	Q	(	P	→	Q	)	¬	(	P	∨	Q	)	¬	P
T	T		T	T	T		F		T	T	T		F	T
T	F		T	F	F		F		T	T	F		F	T
F	T		F	T	T		F		F	T	T		T	F
F	F		F	T	F		T		F	F	F		T	F

Truth Table 3.12 – Truth table test for $P \to Q, \neg(P \vee Q)$ therefore P. Since row 4 is a row where all of the wffs are T, it is possible for the premises and the negation of the conclusion to be true. Therefore, it is possible for the premises to be true and the conclusion false. And so, the truth-table test shows the argument to be invalid.

Exercise 4: *Translate the following arguments into* **PL** *and then use truth-table method to determine whether they are deductively valid or invalid.*

1. John is happy or Mary is hungry. It is not the case that Mary is hungry. Therefore, John is not happy.
2. John will sell his house if and only Mary sells her apartment. Mary will not sell her apartment. Therefore, John will not sell his house.
3. If John sells his house or Mary sells her apartment, the housing market will crash. The housing market will not crash. Therefore, John did not sell his house.

3.3.2 Consistency and Equivalence

We may wish to know whether two propositions are equivalent to each other. That is, we may wish to know whether two statements in English say the same thing (or are true and false under the same conditions). A trivial example is that a proposition is equivalent to itself. For example, "John is tall" and "John is tall" are true and false under the same conditions and so are equivalent. A more complex example might be "If John is tall, then Mary is happy" and "John is not tall or Mary is happy."

In **PL** the members of a set of wffs are equivalent provided all of the members of the set receive the same truth value (T or F) for every interpretation.

DEFINITION – EQUIVALENCE

A pair of wffs P, Q are logically equivalent in **PL** if and only if P and Q receive the same truth value (T or F) under every way of interpreting the propositional letters in P and Q.

A truth table offers a method for testing whether a set of propositions are equivalent.

Definition – Truth Table Test for Equivalence

In a truth table for an equivalence, there is no row on the truth table where one of the pair P has a different truth value than the other Q.

P	Q	(	P	→	Q	)	(	¬	P	∨	Q	)
T	T		T	T	T			F	T	T	T	
T	F		T	F	F			F	T	F	F	
F	T		F	T	T			T	F	T	T	
F	F		F	T	F			T	F	T	F	

Truth Table 3.13 – Truth table for $P \to Q, \neg P \vee Q$. Table shows this wff to a be equivalent since receive the same truth value (T or F) under every way of interpreting the propositional letters.

In **PL** a set of wffs is consistent provided all of the wffs are true under at least one interpretation (valuation).

Definition – Consistency

A set of wffs $\{P, Q, R, \ldots Z\}$ is consistent if and only if there is at least one interpretation of the propositional letters in $P, Q, R, \ldots Z$ such that $P, Q, R, \ldots Z$ are true.

A truth table offers a method for testing whether a set of wffs are consistent.

Definition – Truth-Table Test for Consistency

A truth table shows that a set of propositions is consistent when there is at least one row on the truth table where $P, Q, R, \ldots Z$ are all true.

P	Q	R	(	P	→	Q	)	(	¬	R	∨	Q	)	(	R	∧	¬	Q	)
T	T	T		T	T	T			F	T	T	T			T	F	F	T	
T	T	F		T	T	T			T	F	T	T			F	F	F	T	
T	F	T		T	F	F			F	T	F	F			T	T	T	F	
T	F	F		T	F	F			T	F	T	F			F	F	T	F	
F	T	T		F	T	T			F	T	T	T			T	F	F	T	
F	T	F		F	T	T			T	F	T	T			F	F	F	T	
F	F	T		F	T	F			F	T	F	F			T	T	T	F	
F	F	F		F	T	F			T	F	T	F			F	F	T	F	

Truth Table 3.14 – Truth table for $(P \to Q), (\neg R \vee Q), R \wedge \neg Q$. Table shows this set of wffs to be inconsistent since there is no row where each wff is true.

Exercise 5:

1. $\neg P \wedge \neg R, \neg(P \vee R)$, equivalent?
2. $P, P \to R, R \vee P$, consistent?
3. $P \vee R, \neg R, \neg P$, consistent?
4. Are the following two propositions equivalent: If John is happy, then

Mary is sad. John is not happy or Mary is sad.
5. Is the following set of propositions consistent: John is happy if and only if Mary is sad. John is not happy or Mary is sad. Mary is not sad.
6. $\neg P \to R, R \to \neg P$, test for equivalence
7. $\neg\neg P \to R, R \to Z, Z$, test for consistency
8. $P \vee R, P \to R, P \leftrightarrow R, P \wedge R$, test for consistency

3.3.3 Contingency, Tautology, Contradiction

Definition – Tautology

A proposition P is a tautology (a logically valid formula) if and only if P is true under every valuation.

Definition – Truth-Table Test for Tautology

A truth table for a tautology will have all T's under its main operator (or in the case of no operators, under the propositional letter).

Definition – Contradiction

A proposition P is a contradiction if and only if P is false under every valuation.

Definition – Truth-Table Test for Contradiction

A truth table for a contradiction will have all Fs under its main operator (or in the case of no operators, under the propositional letter).

Definition – Contingency

A proposition P is a contingency if and only if P is neither always false under every valuation nor always true under every valuation.

Definition – Truth-Table Test for Contingency

A truth table for a contingency will have at least one T and at least one F under its main operator (or in the case of no operators, under the propositional letter).

As an illustration, consider $\neg(\neg P \to \neg Q)$. Is this proposition a contradiction, tautology, or contradiction (see Truth Table 3.15)?

P	Q	¬	(	¬	P	→	¬	Q	)
T	T	*F*		*F*	T	T	*F*	T	
T	*F*	*F*		*F*	T	T	T	*F*	
F	T	T		T	*F*	*F*	*F*	T	
F	*F*	*F*		T	*F*	T	T	*F*	

Truth Table 3.15 – Truth table for $\neg(\neg P \rightarrow \neg Q)$. Table shows this wff to a be a contingency since there is at least one T and at least one F under the main operator.

Exercise 6: *Using the truth-table method, determine whether the following wffs are tautologies, contradictions, or contingencies*

1. $\neg P \rightarrow \neg P$
2. $(P \wedge \neg P) \wedge Q$
3. $P \leftrightarrow \neg R$
4. $P \rightarrow (P \vee Q)$
5. $\neg\neg P \wedge P$
6. $\neg(P \vee \neg R)$

3.4 Limitations of Truth Tables

Provided an argument is capable of being fully expressed by a truth-functional language like **PL**, the truth-table method seemingly guarantees there is a way to determine whether that argument is "valid" or "invalid". Since, for example, arguments like that expressed in Argument 3.1 can be expressed in **PL** as $P \rightarrow R, P \models R$, the truth-table method can be used to determine whether the argument is valid.

P1	If Jon looks fishy, then he committed the crime.
P2	Jon looks fishy.
C	Therefore he committed the crime.

Argument 3.1 – An Argument in English

One problem with the truth-table method though is that not every English argument can be represented in a truth-functional language like **PL**. This means that while the truth-table test is an effective tool for determining the validity of *some* arguments we might express in English, it cannot express *all* English arguments. In chapter 6, we articulate a formal language capable of representing non-truth-functional arguments.

Second, from a user's standpoint, the truth-table test is relatively easy to use when dealing with arguments that involve one, two, or even three propositional letters, but they become increasingly complex the more propositional letters the argument has. For example, a truth table for $P \models P$ consists of two rows, one where $v(P) = T$ and one where $v(P) = F$. And, a truth table for $P \models Z$ consists of four rows. The number of rows required for a truth table of any argument is determined by 2^n where n is the number of propositional letters in the argument.

Thus, $P, Q \models Z$ consists of eight rows ($2^3 = 8$), $P, Q, R \models Z$ of sixteen rows ($2^4 = 16$), and so on. Considering that there are arguments composed of dozens of sentences represented by distinct propositional letters, the truth-table test can quickly become unwieldly, requiring hundreds or thousands of rows.

Questions

1. What is a truth table?
2. What is a decision procedure?
3. What does it mean for a wff **Q** to be a semantic consequence of a set of wffs Γ?
4. What does it mean for an argument to be valid and invalid in **PL**?
5. What is the difference between a semantic entailment and a valid argument?
6. Under what conditions does the truth-table method show an argument to be valid? Under what conditions does it show an argument to be invalid?
7. What does it mean to say that two wffs in **PL** are equivalent?
8. Under what conditions does the truth-table method show a pair of wffs to be equivalent?
9. What does it mean to say that a set of wffs in **PL** are consistent?
10. Under what conditions does the truth-table method show a set of wffs to be consistent?
11. What does it mean to say that a wff in **PL** is a tautology, a contradiction, and a contingency?
12. Under what conditions does the truth-table method show that a wff in **PL** is a tautology, a contradiction, and a contingency?
13. What are two problems with using truth tables to check whether an argument expressed in English is valid or invalid?
14. Determine the truth value of $\neg(P \to \neg Q)$ using the following interpretation: $\mathscr{I}(P) = T, \mathscr{I}(Q) = F$
15. Using the truth-table method, determine the truth value of $\neg((P \vee Q) \leftrightarrow Q)$ for each interpretation of P and Q
16. Using the truth-table method, determine whether $Q \to P$ is a semantic consequence of $P \vee \neg Q$. That is determine whether $P \vee \neg Q \models Q \to P$
17. Using the truth-table method, determine whether $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are equivalent.
18. Using the truth-table method, determine whether $P \to Q, Q \to P, P \leftrightarrow Q$ are consistent.
19. Using the truth-table method, determine whether $\neg(P \vee \neg\neg P$ is a tautology, contradiction, or contingency
20. Translate the following propositions from this argument into **PL**, then use the truth-table method to determine whether the argument is valid or invalid: If John is tall or Mary is not happy, then Frank is not tall. Frank is not tall. Therefore, either John is tall or Mary is not happy.