Directions: This exam has 13 questions, for a total of 100 points and 10 bonus points. Place your name on the answersheet (last page). Place proofs on the blank space on the answersheet. Good luck!

Directions: Writing the abbreviation (e.g., $\forall I$) for the single derivation rule that is represented in the following:

1. (2 points) Starting from $(\exists x)Fx$, suppose Fa is assumed. Next, suppose ϕ is derived in the subproof starting with Fa. Finally, suppose ϕ is deprived using $(\exists x)Fx$ and the entire subproof.
2. (2 points) $\neg Qab \land Pa \vdash (\exists z)(\neg Qzb \land Pz)$
3. (2 points) $(\forall z)(Fz \land \neg Fz) \vdash Fd \to \neg Fd$
4. (2 points) From $Qc \land Fc$ to $(\forall y)(Qy \land Fy)$ provided (1) c is not in a premise or in an assumption of an active subproof and (2) c is not in $(\forall y)(Qy \land Fy)$?
5. (2 points) $\neg(\forall z)Fz \vdash (\exists z)\neg Fz$

Directions: Provide proofs for the following:

6. (15 points) $(\exists x)Fx \to (\forall y)By, Fa \vdash Ba$
7. (15 points) $Faa, (\forall x)(\forall y)Lxy \vdash (\exists z)(\exists x)Lzx$
8. (15 points) $(\exists x)\neg Mx \vdash (\exists y)(My \lor \neg Py)$
9. (15 points) $\vdash (\forall x)(Lxx \to (Gx \to Lxx))$
10. (15 points) $\vdash (\forall x)(Lxx \to (Gx \to Lxx))$

Directions: Translate the following arguments and then provide a proof of the conclusion from the premises. In some cases, part of the argument will be provided for you.

11. (15 points) Some Penn State students are good logicians ($(\exists x)(Px \land Gx)$). All good logicians are smart. Therefore, some Penn State students are smart.

Directions: Bonus questions. Totally optional.

12. (9 points (bonus)) Let $=$ be a new operator in QL such that $\alpha = \alpha$ is a wff, where α is a QL name. Now let $=E$ be a derivation rule such that from $\alpha = \beta$ and a wff ϕ containing α (or β), you may substitute α for β in ϕ or β for α. In other words, $a = b, \phi \vdash \phi(a/b)$ or $a = b, \phi \vdash \phi(b/a)$. With this in mind, prove $a = b, b = c, Pa \vdash Pc$.
13. (1 point (bonus)) Free point. If you are reading this, it was very nice to have you in my class.

Derivation rules

Definition 0.0: Conjunction Introduction $\wedge I$

$\phi, \psi \vdash \phi \wedge \psi$ or $\phi, \psi \vdash \psi \wedge \phi$

Definition 0.0: Conjunction Elimination ($\wedge E$)

$\phi \wedge \psi \vdash \phi$ or $\phi \wedge \psi \vdash \psi$

Definition 0.0: Conditional Introduction ($\rightarrow I$)

n	$\quad\phi$	A
	$\quad\vdots$	
m	$\quad\psi$	
m+1	$\phi \rightarrow \psi$	$\rightarrow I$ n-m

Definition 0.0: Conditional Elimination ($\rightarrow E$)

$\phi \rightarrow \psi, \phi \vdash \psi$

Definition 0.0: Reiteration (R)

$\phi \vdash \phi$

Definition 0.0: Negation Introduction ($\neg I$)

k	$\quad\phi$	A
$\vdots$	$\quad\vdots$	
n	$\quad\psi$	
n+1	$\quad\neg(\psi)$	
n+2	$\neg(\phi)$	$\neg I$ k-(n+1)

Definition 0.0: Negation Elimination ($\neg E$)

k	$\quad\neg(\phi)$	A
$\vdots$	$\quad\vdots$	
n	$\quad\psi$	
n+1	$\quad\neg(\psi)$	
n+2	ϕ	$\neg E$ k-(n+1)

Definition 0.0: Disjunction Introduction ($\vee I$)

$\phi \vdash \phi \vee \psi$ or $\phi \vdash \psi \vee \phi$

Definition 0.0: Disjunction Elimination ($\vee E$)

n	$\phi \vee \psi$	P
k	$\quad\psi$	A
	$\quad\vdots$	
k+i	$\quad\chi$	
j	$\quad\psi$	A
	$\quad\vdots$	
j+l	$\quad\chi$	
m	χ	$\vee E$ n, k-(k+i), j-(j+l)

Definition 0.0: Biconditional Introduction ($\leftrightarrow I$)

n	$\quad\phi$	A
	$\quad\vdots$	
n+i	$\quad\psi$	
k	$\quad\psi$	A
	$\quad\vdots$	
k+j	$\quad\phi$	
m	$\phi \leftrightarrow \psi$	$\leftrightarrow I$ n+1, k+j

Definition 0.0: Biconditional Elimination ($\leftrightarrow E$)

$\phi \leftrightarrow \psi, \phi \vdash \psi$ or $\phi \leftrightarrow \psi, \psi \vdash \phi$

Definition 0.0: Disjunctive Syllogism (DS)

$\phi \vee \psi, \neg(\psi) \vdash \phi$ or $\phi \vee \psi, \neg(\phi) \vdash \psi$

Definition 0.0: Modus Tollens (MT)

$\phi \rightarrow \psi, \neg(\psi) \vdash \neg(\phi)$

Definition 0.0: Hypothetical Syllogism (HS)

$\phi \rightarrow \psi, \psi \rightarrow \chi \vdash \phi \rightarrow \chi$

Definition 0.0: Double Negation (DN)

$\phi \dashv\vdash \neg\neg(\phi)$

Definition 0.0: De Morgan's Laws (DeM)

$\neg(\phi \vee \psi) \dashv\vdash \neg(\phi) \wedge \neg(\psi)$ or $\neg(\phi \wedge \psi) \dashv\vdash \neg(\phi) \vee \neg(\psi)$

Definition 0.0: Implication (IMP)

$\phi \rightarrow \psi \dashv\vdash \neg(\phi) \vee \psi$

Definition 0.0: Universal Elimination ($\forall E$)

$(\forall x)\phi(x_1 \ldots x_n) \quad \vdash \quad \phi(\alpha_1 \ldots \alpha_n / x_1 \ldots x_n)$ where x is not in $\phi(\alpha_1 \ldots \alpha_n)$

Definition 0.0: Existential Introduction ($\exists I$)

$\phi(\alpha_i) \vdash (\exists x)\phi(x_n/\alpha_n)$ where x is not in $\phi(\alpha_i)$

Definition 0.0: Universal Introduction ($\forall I$)

$\phi(\alpha_1 \ldots \alpha_n) \quad \vdash \quad (\forall x)\phi(x_1 \ldots x_n/\alpha_1, \ldots \alpha_n)$ where the name α does not occur as premise, as an assumption in an open subproof, or in $(\forall x)\phi(x_1 \ldots x_n/\alpha_1, \ldots \alpha_n)$ and where x is not in $\phi(\alpha_1 \ldots \alpha_n)$

Definition 0.0: Existential Elimination ($\exists E$)

$$
\begin{array}{rlll}
\text{n} & (\exists x)\phi x & & \text{P} \\
\text{j} & \quad \big| \; \underline{\phi(\alpha/x)} & & \text{A} \\
 & \quad \big| \; \vdots & & \\
\text{j+i} & \quad \big| \; \psi & & \\
\text{j+(i+1)} & \psi & & \exists E, \text{ n, j-(j+1)}
\end{array}
$$

Definition 0.0: Quantifier Negation (QN)

$\neg(\forall x)\phi \dashv\vdash (\exists x)\neg\phi$ or $\neg(\exists x)\phi \dashv\vdash (\forall x)\neg\phi$

1. ________________

2. ________________

3. ________________

4. ________________

5. ________________

Solutions for qlproofs/su26a

1. $\exists E$
2. $\exists I$
3. $\forall E$
4. $\forall I$
5. QN
6. $(\exists x)Fx \to (\forall y)By, Fa \vdash Ba$

1	$(\exists x)Fx \to (\forall y)By$	P
2	Fa	P
3	$(\exists x)Fx$	$2\exists I$
4	$(\forall y)By$	$1{,}3\to E$
5	Ba	$4\forall E$

7. $Faa, (\forall x)(\forall y)Lxy \vdash (\exists z)(\exists x)Lzx$

1	Faa	P
2	$(\forall x)(\forall y)Lxy$	P
3	$(\forall y)Lay$	$2\forall E$
4	Laa	$3\forall E$
5	$(\exists x)Lax$	$4\exists I$
6	$(\exists z)(\exists x)Lzx$	$5\exists I$

8. $(\exists x)\neg Mx \vdash (\exists y)(My \lor \neg Py)$

1	$(\exists x)\neg Mx$	P
2	$\quad \neg Ma$	A
3	$\quad \neg Ma \lor \neg Pa$	$2\lor I$
4	$\quad (\exists y)(My \lor \neg Py)$	$3\exists I$
5	$(\exists y)(My \lor \neg Py)$	1, 2-4 $\exists E$

9. $\vdash (\forall x)(Lxx \to (Gx \to Lxx))$

1	$\quad Pa$	A
2	$\quad Pa$	1 R
3	$Pa \to Pa$	1-2$\to I$
4	$(\forall x)(Px \to Px)$	$3\forall I$

10. $\vdash (\forall x)(Lxx \to (Gx \to Lxx))$

1	$\quad Laa$	A
2	$\quad\quad Ga$	A

3	$\quad\vert\quad\vert\ Laa$	1 R
4	$\quad\vert\ Ga \rightarrow Laa$	2-3$\rightarrow I$
5	$Laa \rightarrow (Ga \rightarrow Laa)$	1-4$\rightarrow I$
6	$(\forall x)(Lxx \rightarrow (Gx \rightarrow Lxx))$	5$\forall I$

11. Translation: $(\exists x)(Px \wedge Gx), (\forall x)(Gx \rightarrow Sx) \vdash (\exists x)(Px \wedge Sx)$

1	$(\exists x)(Px \wedge Gx)$	P
2	$(\forall x)(Gx \rightarrow Sx)$	P, $(\exists x)(Px \wedge Sx)$
3	$\quad\vert\ \underline{Pa \wedge Ga}$	A
4	$\quad\vert\ Ga \rightarrow Sa$	2$\forall E$
5	$\quad\vert\ Pa$	3$\wedge E$
6	$\quad\vert\ Ga$	3$\wedge E$
7	$\quad\vert\ Sa$	5,6$\rightarrow E$
8	$\quad\vert\ Pa \wedge Sa$	5,7$\wedge I$
9	$\quad\vert\ (\exists x)(Px \wedge Sx)$	8$\exists I$
10	$(\exists x)(Px \wedge Sx)$	1, 3-9$\exists E$