Chapter 1

MPL Symbols, Syntax, Semantics, Translation

Contents

1.1 MPL Symbols

MPL is an extension of **PL** in that it adds two one-place operators: $\Diamond$ (the diamond) and $\Box$ (the box). The symbols of MPL consist of the following::

1. Uppercase Roman (unbolded) letters $A_1, A_2, B, C, \ldots, Z$ with or without subscripted integers for propositional letters.
2. Truth-functional operators $(\neg, \wedge, \vee, \rightarrow, \leftrightarrow)$
3. Parentheses, braces, and brackets to indicate the scope of operators.
4. Two one-place modal operators ($\Diamond$ and $\Box$).

Exercise 1: State whether the following are symbols in MPL

1. $\Diamond$
2. $\Box$

3. P
4. Q
5. $\neg$
6. 7
7. &

1.2 MPL Syntax

The syntax of **MPL** is identical to **PL** except for well-formed formulas (wff) containing the modal operators: $\Diamond$ and $\Box$. As with **PL** and **RL**, the syntax for **MPL** is expressed using a set of formation rules. An MPL-wff is a formula constructed using the following formation rules:

1. Single propositional letters $A, B, C, \ldots, Z$ with and without numerical subscripts are wffs.
2. If ϕ is a wff, then so is $\neg(\phi)$ is a wff.
3. If ϕ and ψ are wffs, then so are $\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$.
4. If ϕ is a wff, then so are $\Diamond(\phi)$ and $\Box(\phi)$.
5. Nothing else is a wff except in virtue of (1)–(4).

To improve readability, parentheses are dropped when there is no confusion about the scope of an operator or in cases where the monadic operators are iterated. Thus, rather than $\Box(\Diamond(\phi))$, we write $\Box\Diamond\phi$. In addition, rather than $\neg(\Diamond(\phi))$, we write $\neg\Diamond\phi$.

All of these are MPL wffs:

1. P
2. $P \wedge Q$
3. $\neg P$
4. $\Diamond P$
5. $\Diamond\Diamond P$
6. $\neg\Diamond P$
7. $\neg\Diamond\neg\Diamond P$
8. $\Box P$
9. $\Box\Box P$
10. $\neg\Box P$
11. $\neg\Box\neg\Box P$
12. $\Box\Diamond P$
13. $\Box(\Diamond P \rightarrow P)$

All of these are not MPL wffs:

1. PP
2. $\wedge Q$
3. $P\neg$
4. $P\Diamond$
5. $\Diamond P\Diamond$
6. $\Diamond\Diamond$
7. $P\Diamond Q$

Exercise 2: Use the formation rules to show that the following are MPL-wffs

1. $\Box P$
2. $\neg\Box P$
3. $P \leftrightarrow \Box Q$

4. $\Diamond\Box Q$
5. $\Box\Diamond P$
6. $\Diamond\Box P \vee \Box Q$
7. $\Box\Box\Diamond Q$
8. $\Box\neg\Diamond Q$
9. $\Diamond\Diamond\Box\neg Q$
10. $(\Box P \rightarrow \Diamond\neg Q) \vee \Diamond S$

1.3 MPL Semantics

The semantics of MPL are determined using a model M.

Definition 1 MPL *Model An* **MPL** *model is a ordered triple* $\langle \mathcal{W}, \mathcal{R}, \mathscr{I} \rangle$ *consisting of the following:*

1. *$\mathcal{W}$, a non-empty set of objects (or "worlds"). That is, $w_1, w_2, \ldots w_n \in W$.*
2. *$\mathcal{R}$, a binary relation over $\mathcal{W}$. That is, $R \subseteq W \times W$.*
3. *$\mathscr{I}$, a 2-place function that takes pairs of propositional letters and worlds (e.g. P, w_1) as input and outputs a truth value T or F. That is, $\mathscr{I} : (P \times W) \mapsto \{F, T\}$ where P is the set of propositional letters.*

The MPL model is a three-part structure, but it also can be thought of in terms of a two-part structure: $\langle \mathcal{F}, \mathscr{I} \rangle$ where $\mathcal{F}$ is the model's frame and $\mathscr{I}$ is an interpretation function. The frame $\mathcal{F}$ is a tuple $\langle W, R \rangle$ consisting of a **set of possible worlds** W and a relation $\mathcal{R}$ on $\mathcal{W}$ called the **accessibility relation**. A MPL-model can be divided into two parts: the frame $\mathcal{F}$ of the model and the interpretation function $\mathscr{I}$. The frame of a model $\mathcal{M}$ is the ordered pair $\mathcal{W}$ and $\mathcal{R}$.

1.3.1 Possible Worlds

The part of a model is a non-empty set $\mathcal{W}$. This set is sometimes said to consist of "possible world", designated by w with subscripted numbers or simply numbers. That is, the elements of $\mathcal{W}$ are represented using lower case w with positive subscripted integers. For example, we can specify that the set of worlds $\mathcal{W}$ consists of three worlds w_1, w_2, w_3 by writing $w_1, w_2, w_3 \in \mathcal{W}$.

The definition, nature, and reality of possible worlds is the subject of philosophical dispute. Some people think that there is no such thing as possible worlds, other people think they are real but abstract, while still others think they are real and just as real as this world. We will avoid these metaphysical issues here and simply define a "possible world" (represented by w) to be a complete (maximally inclusive) scenario, viz., it is a fully specific way in which a world can be. In other words, these worlds are defined in such a way that for any proposition p, that proposition p is determinate enough for p to be either true or false.

1.3.2 Interpretation Function

The second part of an MPL model is the interpretation function. The interpretation function $\mathscr{I}$ is a 2-place function that takes propositional letters at worlds in $\mathcal{W}$ as input and delivers a truth value T, F as output. The $\mathscr{I}$ behaves the same as the PL-interpretation function except that it assigns truth values to propositional letters at worlds. Thus, $\mathscr{I}$ does not merely take $A, B, C, \ldots Z$ as input and deliver truth values as output. It instead takes A at w_1 and A at w_2 and B at w_1, and so on and delivers truth values as output.

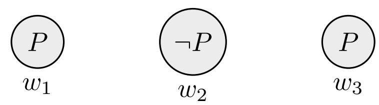
We can specify the interpretation of a proposition at a possible world in at least three ways. First, we can simply write out the function. For example, if P is a propositional letter, we can specify the worlds at which it is true and false as follows:

- $\mathscr{I}(P, w_1) = T$
- $\mathscr{I}(P, w_2) = F$
- $\mathscr{I}(P, w_3) = T$

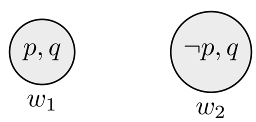
Second, we can also list the worlds and then specify whether P is true or false. If P is true, then we can write P. If P is false, then we can write $\neg P$. Here is an example mirroring the above interpretation of P at worlds w_1, w_2, w_3:

- $w_1 : P$
- $w_2 : \neg P$
- $w_3 : P$

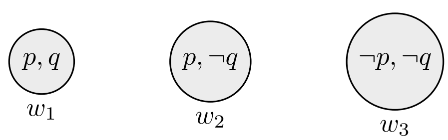
Finally, it is common to represent the interpretation of propositions using a graph (or diagram).



Let's look at another example. Suppose we had two possible worlds w_1 and w_2 and two propositional letters p, q. We could specify the truth values of p and q at each world. We would do this by writing p if $v(p) = T$ and $\neg p$ if $v(p) = F$.



In the above example, p, q are true at w_1 while $\neg p, q$ are true at w_2. Alternatively, consider the following example:



In the above example, p, q are true at w_1, $p, \neg q$ are true at w_2, and $\neg p, \neg q$ are true at w_3.

Exercise 3

Create a graph for the following partial models.

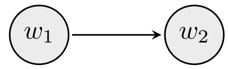
1. $w_1 : p, q, r, w_2 : \neg p, \neg q, r,$
2. $w_1 : p, q, r, w_2 : p, \neg q, r, w_3 : p, q, \neg r$
3. $w_1 : p, w_2 : \neg p, w_3 : p, w_4 : \neg p$

1.3.3 Accessibility Relation

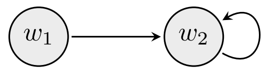
We have discussed the set of worlds W and the interpretation function $\mathscr{I}$. The last part of an MPL model is the accessibility relation $\mathcal{R}$. The accessibility relation $\mathcal{R}$ is a binary relation on $\mathcal{W}$ ($R \subseteq W \times W$). It thus consists of a set of ordered pairs $\langle w_1, w_2 \rangle$ where both elements are from W. The $\mathcal{R}$ is usually referred to as an "accessibility relation". When $\langle w1, w2 \rangle \in R$, w_1 is said to "access" w_2.

The accessibility relation is specified in one of two ways. First, the ordered pairs may be enumerated $\langle w_1, w_1 \rangle, \langle w_1, w_2 \rangle$ or written as a relation: $\mathcal{R}w_1w_2$. The ordered pair $\langle w_1, w_2 \rangle$ and relation $\mathcal{R}w_1w_2$ both can be read as saying that w_1 accesses w_2 (this should not be interpreted as w_1 and w_2 can access each other).

Second, we can represent the accessibility relation using a graph. Let's suppose a simple accessibility relation involving two worlds w_1 and w_2 where w_1 accesses w_2:

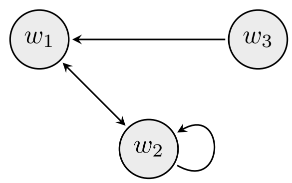


The arrow is used to indicate that w_1 accesses w_2. Let's consider another example involving two worlds and the relation $\mathcal{R}w_1w_2, \mathcal{R}w_2w_2$:



In the above example, w_1 accesses w_2 and w_2 access w_2, but notice that w_1 does not access w_1 nor does w_2 access w_2.

Finally, let's consider an example involving three worlds where $\langle w_1, w_2\rangle, \langle w_2, w_1\rangle, \langle w_2, w_3\rangle, \langle w_2, w_2\rangle, \langle w_3, w_1\rangle$:



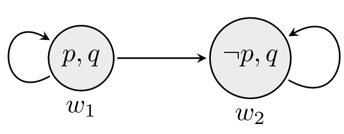
The basic idea behind the accessibility relation is that it expresses what is possible relative to a specific world. So, if we write $\mathcal{R}w_1w_2$, we are saying that "the propositions in world w2 are possible relative to world w1."

1.3.4 Graphs for Models

Recall that a model consists of a set of possible worlds (W), an interpretation of propositional letters at those worlds ($\mathscr{I}$), and an accessibility relation. The possible worlds, the interpretation of propositional letters at those worlds, and the accessibility relation can all be represented using one graph. Suppose there are two worlds w_1 and w_2. Also suppose the following:

- $w_1 : p, q$
- $w_2 : \neg p, q$
- $\langle w_1, w_1\rangle, \langle w_2, w_2\rangle, \langle w_1, w_2\rangle$

With this model in place, we can represent the entirety using a graph:



Exercise 4

Create a graph for the following models.

1. $w_1 : p, q, w_2 : p, \neg q; \langle w_1, w_1\rangle$
2. $w_1 : p, q, w_2 : p, \neg q; \langle w_1, w_1\rangle, \langle w_2, w_2\rangle$
3. $w_1 : p, \neg q, r, w_2 : p, \neg q, r; \langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, w_2\rangle$

1.3.5 Valuation of wffs relative to models

In PL, a valuation function takes pairs of truth values and worlds as input and delivers a truth value as output. For convenience, the valuation function is often said to take MPL-wffs and worlds as input and deliver a truth value as output. That is, $v_{MPL} : (\phi_{MPL} \times W) \mapsto \{F, T\}$, where ϕ_{MPL} is the set of MPL-wffs.

Definition 2 *Valuation Function When $\mathcal{M}$=$\langle\mathcal{W}, \mathcal{R}, \mathscr{I}\rangle$, a valuation of $\mathcal{V}_{\mathcal{M}}$ is a two-placed function that assigns either T or F (not both) to every wff relative to a world in $\mathcal{W}$ in the following manner (where A is any propositional letter while ϕ and ψ refer to any MPL-wff:*

1. $\mathcal{V}_{\mathcal{M}}\ (A, w) = \mathscr{I}(A, w)$.
2. $\mathcal{V}_{\mathcal{M}}\ (\neg(\phi), w) = T$ *iff* $\mathcal{V}_{\mathcal{M}}\ (\phi, w) = F$.
3. $\mathcal{V}_{\mathcal{M}}\ ((\phi \wedge \psi), w) = T$ *iff* $\mathcal{V}_{\mathcal{M}}(\phi, w) = T$ *and* $\mathcal{V}_{\mathcal{M}}(\psi, w) = T$.
4. $\mathcal{V}_{\mathcal{M}}\ ((\phi \vee \psi), w) = T$ *iff* $\mathcal{V}_{\mathcal{M}}(\phi, w) = T$ *or* $\mathcal{V}_{\mathcal{M}}(\psi, w) = T$.
5. $\mathcal{V}_{\mathcal{M}}\ ((\phi \rightarrow \psi), w) = T$ *iff* $\mathcal{V}_{\mathcal{M}}(\phi, w) = F$ *or* $\mathcal{V}_{\mathcal{M}}(\psi, w) = T$.
6. $\mathcal{V}_{\mathcal{M}}\ ((\phi \leftrightarrow \psi), w) = T$ *iff* $\mathcal{V}_{\mathcal{M}}(\phi, w) = T$ *and* $\mathcal{V}_{\mathcal{M}}(\psi, w) = T$ *or* $\mathcal{V}_{\mathcal{M}}(\phi, w) = F$ *and* $\mathcal{V}_{\mathcal{M}}(\psi, w) = F$.
7. $\mathcal{V}_{\mathcal{M}}\ (\Box(\phi), w) = T$ *iff for each* $u \in \mathcal{W}$, *if* $\mathcal{R}wu$, *then* $\mathcal{V}_{\mathcal{M}}\ (\phi, u) = T$.
8. $\mathcal{V}_{\mathcal{M}}(\Diamond(\phi), w) = T$ *iff for at least one* $u \in \mathcal{W}$, $\mathcal{R}wu$ *and* $\mathcal{V}_{\mathcal{M}}(\phi, u) = T$.

To simplify, we exclude subscripting the model to the valuation function. Thus, instead of writing $v_{\mathcal{M}}(\phi) = T$, we simply write $v(\phi) = T$. Using a model and the valuation function, the truth value of every MPL wff can be determined in a step-by-step fashion.

Example 1.1

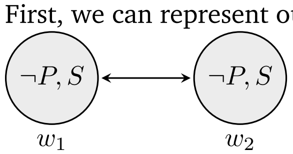
Consider: $\mathcal{M} = \langle\mathcal{W}, \mathcal{R}, \mathscr{I}\rangle$, $\mathcal{W} = \{w_1, w_2\}$, $\mathcal{R} = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$, $\mathscr{I}(P, w_1) = F, \mathscr{I}(P, w_2) = F, \mathscr{I}(S, w_1) = T, \mathscr{I}(S, w_2) = T$. Determine the truth value of $v(\Box P \wedge S, w_1)$.

1	$v(P, w_2) = F$ since $\mathscr{I}(P, w_2) = F$	Rule 1
2	$v(S, w_1) = T$ since $\mathscr{I}(S, w1) = T$	Rule 1
3	$v(\Box P, w_1) = F$ since $\mathcal{R} = \langle w1, w2\rangle$ and $(P, w_2) = F$	1, Rule 6
4	$v(\Box P \wedge S, w1) = F$ since $v(\Box P, w1) = F$ and $v(S, w1) = T$	2,3, Rule 2

Rather than using the valuation rules in such a formulaic way, it tends to be easier to determine the truth value of MPL wffs using graphs. First, let's consider the same example that we considered above, except let's consider it using the graph.

Example 1.2

First, we can represent our model using a graph:

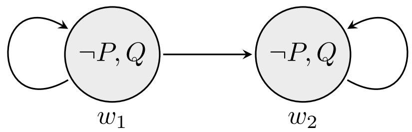


Next, using the above, model, we can determine the truth value of $(\Box P \wedge S, w_1)$. Since S is true at w_1, the right conjunct is true. However, since P is not true at every world accessible to w_1, $\Box P$ is false. Therefore, $v(\Box P, w1) = F$

Let's consider another example. This time we will determine the truth values of $\Diamond P, w_1$ and $\Box Q, w_2$.

Example 1.3

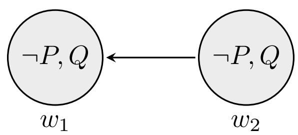
First, we can represent our model using a graph:



For $\Diamond P, w_1$, we check whether there is at least one world accessible to w_1 where P is true. Notice however, there is no such world so $v(\Diamond P, w_1) = F$. For $\Box Q, w_2$, we check whether Q is true for every world accessible to w_2. Since w_2 is only accessible to itself, and Q is true, $v(\Box Q, w_2) = T$.

Let's consider two final cases. This time we will consider $\Diamond P, w_1$ and $\Box P, w_1$ but where w_1 is not accessible to any world.

Example 1.4



For $\Diamond P, w_1$, we check whether there is at least one world accessible to w_1 where P is true. Notice however, there is no such world so $v(\Diamond P, w_1) = F$. Notice that $\Box P, w_1$ is true if and only if for every world u in W, if w_1 accesses u, then P is true in u. But since w_1 does not access any world, it follows that $v(\Box P, w_1) = T$.

Exercise 5

First, create a graph for the following model. Then use the graph to determine whether the following wffs are T or F given the model. $\mathcal{M} = \langle W, R, I \rangle$ where $\mathcal{W} = w_1, w_2$, $\mathcal{R} = \langle w_1, w_2 \rangle, \langle w_1, w_1 \rangle$, and $\mathscr{I}(P, w_1) = T, \mathscr{I}(P, w_2) = F, \mathscr{I}(S, w_1) = T, \mathscr{I}(S, w_2) = T$

1. $v(P \wedge S, w_2)$
2. $v(P \wedge S, w_1)$
3. $v(\Box P, w_1)$
4. $v(\Diamond P, w_1)$
5. $v(\Diamond P, w_2)$
6. $v(\Box S, w_1)$
7. $v(\Box S, w_2)$
8. $v(\Diamond \Box P, w_1)$
9. $v(\Box \Box P, w_1)$
10. $v(\Diamond \Box P, w_1)$

1.4 MPL Translation

The box and diamond of MPL can be translated as "it is necessary" and "it is possible" respectively. Thus,

1. It is necessary that **P**.
2. **P** must be the case.
3. **P** has to be the case.

(1)-(3) all translated as $\Box P$.

4. It is possible that **P**
5. It may be **P**
6. **P** can be the case

(4)-(6) are translated as $\Diamond P$. More complex translations are certainly possible. Consider the following sentences:

7. It is impossible that **P** is the case.
8. It is not necessarily the case that **P** is the case.
9. It must be possible that **P** is the case.
10. It must be necessary that **P** is the case.

To say that something is "impossible" is to say that it is "not possible" and thus (7) is translated as $\neg\Diamond P$. Individuals will often deny something is necessarily the case, and so the translation of (8) is $\neg\Box P$. In the case of (9), it is asserted that P is not merely possible but is necessarily possible. Thus, (9) is translated as $\Box\Diamond P$. Finally, (10) says that P is not merely necessarily the case but it is necessary that P is necessary. Thus, (10) is translated as $\Box\Box P$.

In addition, we can translate various English arguments involving possibility and necessity.

11. It is possible that God exists. Therefore, God exists.
12. It is necessary that God exists. Therefore, God exists.
13. It is either possible or not possible that God exists. Therefore, God exists.

(11) can be translated as $\Diamond G \models G$, (11) as $\Box G \models G$, and (13) as $\Diamond G \vee \neg\Diamond G \models G$.

Finally, it is worth pointing out that some English sentences are ambiguous. For example, consider the sentence "If Tek is a bachelor, then Tek must be unmarried." On the surface, it looks as though the modal operator ($\Box$) should apply to the proposition "Tek is unmarried." And so, if we give this sentence a straightforward translation, it would look as follows:

14. $B \rightarrow \Box\neg M$

However, if this sentence is true, then if Tek is a bachelor at w_1, then he is unmarried at every world accessible to w_1. But this cannot be what is expressed by the meaning of this sentence since we can imagine that Tek is a bachelor at this world, but married at another world accessible to this one. More simply, just because Tek is a bachelor here and now doesn't mean that he's not married at some other world. Rather than affixing the modal operator to the consequence of the consequent, we capture the meaning of the sentence by giving the modal operator wide scope. A better translation is the following:

15. $\Box(B \rightarrow \neg M)$

Again, consider a world w_1, This says that for every world accessible to w_1 is a world such that if Tek is a bachelor, then Tek is married. This captures what we mean for it says that if you find a world where Tek is a bachelor, that is also a world where he is unmarried.

Exercise 6

Translate the following sentences into MPL. Use the following key: G: God exists

1. God exists
2. It is possible that God exists
3. It is necessary that God exists
4. God exists if and only if it is possible that God exists
5. If it is possible that God exists, then it is necessary that God exists.
6. It is not impossible that God exists.
7. It must be possible that God exists
8. If it is impossible that God exists, then God must not exist.
9. If God exists, then it must be possible that God exists.

10. If it is necessary that God exists, then it must be possible that God exists.

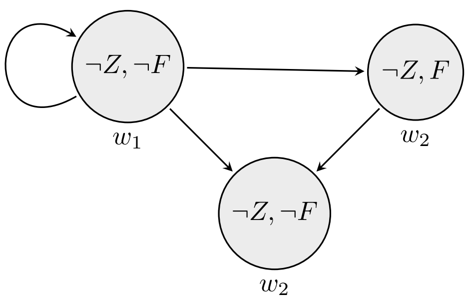
Exercise 7

Translate the following arguments into MPL. Use the following key: G: God exists. Use $\models$ to translate "therefore".

1. God exists. Therefore it is possible that God exists.
2. It is possible that God exists. Therefore, God exists.
3. If it is possible that God exists, then God exists. It is possible that God exists. Therefore, God exists.
4. It is impossible that God exists. Therefore, God does not exist.

Exercise 8

Use the model below and the follow key to determine the truth of each sentence. Key: Z: Zombies eat brains. F: Pegasus flies. Use the following model to determine if the sentence below are true or false. Evaluate the truth or falsity of each sentence at w_1.



1. It is possible that a Pegasus flies.
2. It is necessarily the case that a Pegasus flies.
3. It is possible that Zombies eat brains.
4. It is possible that Liz exists and zombies eat brains.
5. It is necessarily the case that if a zombie eats brains, then a Pegasus flies.

1.5 Validity

In this section, two different notions of a "valid-wff in MPL" are formulated (also known as a tautology).

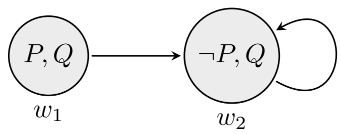
1.5.1 Validity in a model

First, a wff ϕ may be **valid in a model**.

Definition 3 *valid-in-a-model A MPL-wff* ϕ *is valid-in-a-model* $\mathcal{M}$ *where* $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathscr{I} \rangle$ *iff for every* $w \in \mathcal{W}, v_M(\phi, w) = T$.

In other words, if a wff ϕ is true in each of a model's worlds, then ϕ is valid in that model. Take the wff $P \to P$ for the model $\mathcal{M}_1$.

Example 1.5: $\mathcal{M}_1$

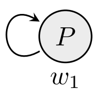


At w_1, since $v(P) = T$, it follows that $v(P \to P) = T$. At w_2, since $v(P) = F$, it follows that $v(P \to P) = T$. Since $v(P \to P) = T$ at every world in M, it follows that $P \to P$ is valid in the model $\mathcal{M}$.

Now consider the wff $\Box P \to P$ for the same model M. At w_1, since $w_1 R w_2$ and since $v(P, w_2) = F$, it follows that $v(\Box P) = F$ and so $v(\Box P \to P) = T$. At w_2, since $w_2 R w_2$ and since $v(P, w_2) = F$, it again follows that $v(\Box P) = F$ and so $v(\Box P \to P) = T$. And so, since $v(\Box P \to P) = T$ at every world in M, it follows that $\Box P \to P$ is also valid in M.

It is important to note that a wff ϕ being valid-in-a-model does not mean it is **valid in every model** for there are some wffs that are valid in one model but not valid in another model. For example, consider the wff P for model below.

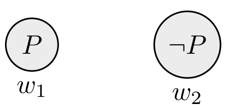
Example 1.6: $\mathcal{M}$



Notice that P is valid in this model.

However, notice that P is not valid in the following model.

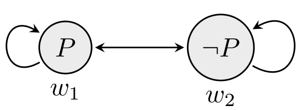
Example 1.7: $\mathcal{M}$



Notice that P is not valid.

Let's illustrate this point with another example. Consider the wff $\Diamond\phi \wedge \Diamond\neg\phi$. That is, it is possible that P is the case and it is not possible that P is the case. First, consider model $\mathcal{M}_2$:

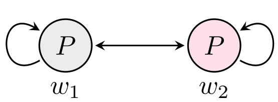
Example 1.8: $\mathcal{M}_2$



Since $v(\Diamond P \wedge \Diamond\neg P) = T$ in both w_1 and w_2, $\Diamond P \wedge \Diamond\neg P$ is valid in $\mathcal{M}_2$.

However, consider this same wff in a different model $\mathcal{M}_3$:

Example 1.9: $\mathcal{M}_3$

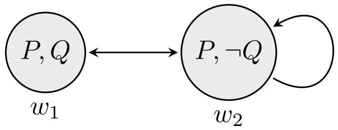


Note that in $w1$, $v(\Diamond P \wedge \Diamond\neg P) = F$ since the right conjunct is false. That is, there is no world that w_1 accesses such that $v(\Diamond\neg P) = T$. Thus, $\Diamond P \wedge \Diamond\neg P$ is invalid in $\mathcal{M}_3$.

What is evident from the previous example is that the whether a wff ϕ is valid-in-a-model depends upon the *interpretation of the model*. That is, you can have two different models with the same frame (set of worlds and accessibility relation), but a wff ϕ can be valid in one and invalid in another.

Exercise 9

Determine whether a wff is valid or invalid in the following model:



1. P
2. $P \wedge Q$
3. $\Diamond Q$
4. $\Diamond P$
5. $\Box P$

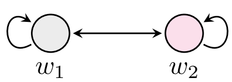
1.5.2 Validity in a frame

The second notion of validity is validity-in-a-frame. In contrast to validity-in-a-model where validity depends upon the interpretation of the model, a wff ϕ is valid-in-a-frame if and only if it is true for **any interpretation** in a frame. In other words, if we say that a wff ϕ is valid-in-a-frame, we are saying that ϕ is true for *any* interpretation in that frame. That is, whether or not it is valid does not depend upon the interpretation of the model. The notion of validity-in-a-frame is sometimes referred to as being **MPL-valid for a system of logic**.

Definition 4 *valid-on-a-frame For a frame δ, a MPL-wff ϕ is frame-valid (where δ is K, T, B, S_4, S_5) iff ϕ is valid in every δ-model.*

As we saw above, the wff $\Diamond\phi \wedge \Diamond\neg\phi$ is not valid in the frame common to both of their models. However, consider the wff $\Box\phi \rightarrow \phi$..

Example 1.10: $\mathcal{M}_4$

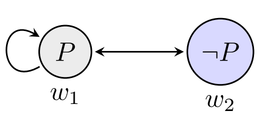


With this frame, we see that the wff $\Box P \rightarrow P$ is frame-valid.

w_1	w_2	$\Box P \rightarrow P, w_1$	$\Box P \rightarrow P, w_2$
P	P	T	T
P	$\neg P$	T	T
$\neg P$	P	T	T
$\neg P$	$\neg P$	T	T

And so while the validity of some wffs depend upon the interpretation of the model (e.g. $\Diamond P \wedge \Diamond\neg P$), the validity of some wffs are independent of the interpretation of the model. It is important to note that the validity of $\Box\phi \rightarrow \phi$ depends upon the **frame** of the model. This wff is not valid in other frames.

Example 1.11: $\mathcal{M}_5$



In $\mathcal{M}_4$, notice that in w_2, $v(\Box P) = T$ but $v(P) = F$.

1.5.3 Modal Logic Frames

Thus far, the models and the frames within the models have been formulated in an arbitrary way. There are, however, properties that characterize a whole class of frames. For example, a set of frames may be characterized as all those having the property of being *reflexive*. That is, for all worlds in the frame, those worlds are accessible to themselves. Or, we might consider the class of frames where the accessibility relation is transitive. That is, for all worlds in the model, if $\langle w_1, w_2 \rangle$ and $\langle w_2, w_3 \rangle$, then $\langle w_1, w_3 \rangle$.

Some wffs will be valid whenever a model has one of these properties. That is, there are some wffs ϕ such that whenever the frame $\mathcal{F}$ has a transitive accessibility relation, then ϕ will be valid-in-the-frame.

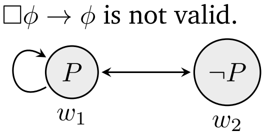
One example is that $\Box\phi \rightarrow \phi$ is valid-in-a-frame if and only if $\mathcal{R}$ is reflexive. To show this, it is necessary to show two propositions. First, it is necessary to show that if a model has a reflexive, then $\Box\phi \rightarrow \phi$ is valid-in-the-frame. Second, one must show that if $\Box\phi \rightarrow \phi$ is valid-in-a-frame, then $\mathcal{R}$ is reflexive.

First, suppose $\mathcal{M}$ has a frame $\mathcal{F}$ with the accessibility relation $\mathcal{R}$ that is reflexive. If $\Box\phi$ is false at every world, then $\Box\phi \rightarrow \phi$ is true at every world. If $\Box\phi$ is true at a world w, then $v(\phi) = T$ at every world w_i accessible to w. But since $\mathcal{R}$ is reflexive, then $\langle w, w \rangle$. And so, $v(\phi, w_i) = T$. Therefore, whenever $\mathcal{R}$ is reflexive, then $\Box\phi \rightarrow \phi$ is valid-in-the-frame.

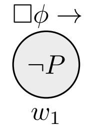
Second, if $\Box\phi \rightarrow \phi$ is valid, then the frame is reflexive. This can be shown by demonstrating that a counterexample to $\Box\phi \rightarrow \phi$ can be constructed whenever the frame is not reflexive. So, suppose a frame $\mathcal{F}$ where the accessibility relation $\mathcal{R}$ is not reflexive. In such a frame, suppose $v(\phi, w) = F$, and $v(\phi, w_i) = T$ for all other worlds w_i. It would thus follow that $v(\Box\phi \rightarrow \phi, w) = F$ and so would not be valid-in-the-frame.

Example 1.12: $\Box\phi \rightarrow \phi$

First example of a non-reflexive frame where $\Box\phi \rightarrow \phi$ is not valid.



Second example of a non-reflexive frame where $\Box\phi \rightarrow \phi$ is not valid.



There is thus a relationship between the validity of a wff ϕ and the properties of a frame $\mathcal{F}$.

1.5.4 Systems of logic

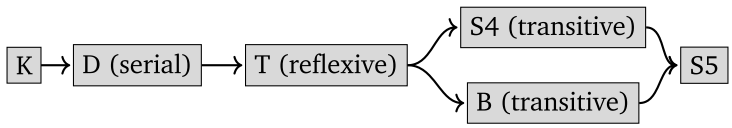
Given the

An S-model for a modal system S is an MPL-model $\langle \mathcal{W}, \mathcal{R}, \mathscr{I} \rangle$ where the accessibility relation $\mathcal{R}$ has the following property or properties:

- K: no requirement on $\mathcal{R}$
- D: $\mathcal{R}$ is serial
- T: $\mathcal{R}$ is reflexive
- B: $\mathcal{R}$ is reflexive and symmetric
- S4: $\mathcal{R}$ is reflexive and transitive
- S5: $\mathcal{R}$ is reflexive, symmetric, and transitive

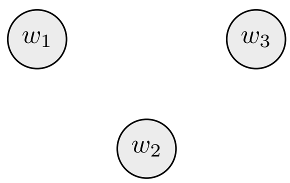
Since there is no requirement on R for system K, it is the least restrictive. This system may consist of models that are serial or not, reflexive or not, transitive or not, and so on. System K is also the *weakest* modal system in that anything that is true of system K will also be true of all stronger (more restrictive) modal systems. For instance, if system K permits transitive and non-transitive accessibility relations and a wff ϕ is true of system K, then it will also be true of all the more restrictive systems where the accessibility relation must be transitive. But note that the reverse is not the case. That is, a wff ϕ may be valid in a more restrictive (stronger) system but not true in system K.

Since K is the weakest of all modal systems, it can be viewed it as foundational. That is, other modal systems may be built from system K simply by placing restrictions on the accessibility relation.[1]



Let's consider some illustrations of these systems. Let's begin with System K.

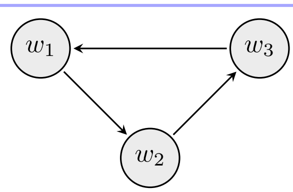
Example 1.13: System K



No requirements on R.

More restrictive (stronger) systems can be built up on System K by placing conditions on the accessibility relation $\mathcal{R}$. In the case of system D, the restriction is that the accessibility relation is serial.

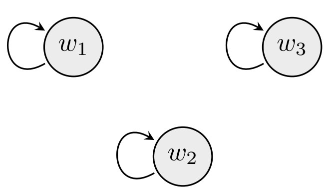
Example 1.14: System D



R is serial: for every world w in $\mathcal{W}$, there is at least one world w_i such that $\langle w, w_i \rangle$.

System T is built by making $\mathcal{R}$ reflexive.

Example 1.15: System T



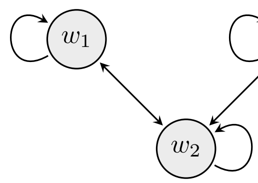
R is reflexive: for every world w in $\mathcal{W}$, $\langle w, w \rangle$.

Note that all reflexive relations are serial but not all serial relations are reflexive. This is because when $\mathcal{R}$ is reflexive, there is at least one world it accesses, namely itself. However, when a relation is serial, it accesses at least one world, but not necessarily itself.

Next, system B may be constructed by stipulating that $\mathcal{R}$ is not only reflexive (and therefore serial) but also symmetric. In other words, system B is built on system T by adding the symmetric restriction.

[1] Some authors start the other way around. That is, one might begin with the strongest, most restrictive system, the one that makes a number of assumptions about $\mathcal{R}$, and then proceed to identify weaker systems by loosening the requirements on $\mathcal{R}$.

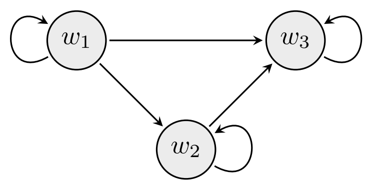
Example 1.16: System B



$\mathcal{R}$ is reflexive, serial, and symmetric. The accessibility relation $\mathcal{R}$ is symmetric iff if $\langle w_i, w_j \rangle$, then $\langle w_j, w_i \rangle$.

Next, system S4 is constructed by stipulating that $\mathcal{R}$ is not only reflexive (and therefore serial) but also transitive. In other words, system S4 is also built on system T by adding the transitive restriction.

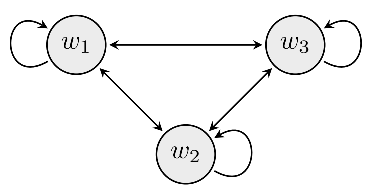
Example 1.17: System S4



$\mathcal{R}$ is reflexive, serial, and transitive. The accessibility relation $\mathcal{R}$ is transitive iff on the condition that $\langle w_i, w_j \rangle$ and $\langle w_j, w_k \rangle$, then $\langle w_i, w_k \rangle$.

Finally, system S5 is constructed by stipulating that $\mathcal{R}$ is not only reflexive (and therefore serial) but also symmetric and transitive. In other words, system S5 may be thought of as being built on system S4 by adding the symmetric restriction or system B by adding the transitive restriction.

Example 1.18: System S5



$\mathcal{R}$ is reflexive, serial, symmetric, and transitive.

Note that S5 places the most restrictions on the accessibility relation since every world is accessible to itself and to every other world.

- $\Box\phi \rightarrow \phi$ is valid whenever $\mathcal{R}$ is reflexive.
- $\Box\phi \rightarrow \Box\Box\phi$ is valid whenever $\mathcal{R}$ is transitive.
- $\Diamond\Box\phi \rightarrow \phi$ is valid whenever $\mathcal{R}$ is symmetric.

1.6 Systems of modal propositional logic

1.7 MPL trees

$$\begin{array}{ll} \neg\Box\phi, w & \\ \Diamond\neg\phi, w & MN \end{array} \qquad\qquad \begin{array}{ll} \neg\Diamond\phi, w & \\ \Box\neg\phi & MN \end{array}$$

$$\begin{array}{ll} \Box\phi, w & \\ wRu & \\ \vdots & \\ \phi, u & \Box K \end{array} \qquad\qquad \begin{array}{ll} \Diamond\phi, w & \\ \vdots & \\ wRu & \Diamond K \\ \phi, u & \Diamond K \end{array}$$

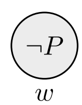
(a) where u is new to the branch

(b) where it has been established that wRu

To illustrate, consider the wff $\Box P \rightarrow P$.

1.	$\neg(\Box P \rightarrow P), w$ ✓	P
2.	$\Box P, w$	1MN
3.	$\neg P, w$	1MN

The branch cannot be closed. The following countermodel would make $\Box P \rightarrow P, w$ false:



Since for every world w, if w accesses that world, then $v(P) = T$, it follows that $v(\Box Pw) = T$. But since $v(P, w) = F$ and so $v(\Box P \rightarrow P, w) = F$. And so $\Box P \rightarrow P$ is not K-valid.