PHIL012 – Symbolic Logic

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Logic refers to the study or good and bad arguments. Deductive logic refers to the study of arguments where the conclusion of an argument is a "logical consequence" of the premises. And, symbolic logic refers to the articulation and study of logic using a formal (symbolic) language. In this course, students will be introduced to two symbolic languages: the language of propositional logic and the language of first-order predicate logic. Students will learn the symbols, syntax (grammar), and semantics of these languages, as well as decision procedures (tables and trees) and proof systems.

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1.1 Introduction: extensions, deviations, variations

A **formal language** is defined as a set of symbols and a syntax (characters and rules for putting characters into well-formed formulas). A **logical system** is a formal language along with a semantics (that is, an interpretation of the symbols and semantics) and a proof-system (a set of derivation rules). The language of propositional logic and the language of predicate logic form systems of logic once they are given a semantics and a proof system.

The systems of propositional logic and predicate logic (as they have been discussed in earlier lessons) are referred to as classical or standard logic. Referring to them as "standard" or "classical" is not meant to indicate that they are better or worse than other logical systems. Rather, these designations are simply useful as reference points for characterizing the panoply of logical systems. In other words, other logical systems can be seen in one of three ways in relation to classical logic:

1. an extension of classical logic
2. a deviation from classical logic
3. a syntactic variation of classical logic

An **extension** of classical logic expands upon or adds to the classical logic systems by adding additional logical constants (operators), which subsequently (but not always) get incorporated into the syntax, semantics, and proof-system of the logical system. A simple example of an extension of classical propositional logic would be to add an additional truth-functional operator ($\uparrow$) to the existing set ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$), then add a syntactic rule to the language (e.g. if ϕ and ψ are wffs, then $\phi \uparrow \psi$ is a wff), add a semantic rule for wffs where $\uparrow$ is the main operator (e.g. $v(\phi \uparrow \psi) = T$ iff $v(\phi) = F$ or $v(\phi) = F$, and then a derivation rule (e.g. $\phi \uparrow (\psi \rightarrow \chi), \phi \vdash \chi$).

Extensions may be divided into **substantive** and **superficial** extensions. A substantive extension increases the expressivity of the logical system: the logical system is now able to capture a semantic or syntactic entailment that it was previously incapable of expressing. The extension of propositional logic to predicate logic is an example of such an extension. Since the language of propositional logic treats propositions as whole units, it is incapable of expressing the internal (subsential) logical structure that is at least partially represented by predicate logic. A superficial extension (like the example involving $\uparrow$ given above) makes no contribution to the expressivity of the logical system. These extensions are often made for convenience, simplicity, or for specific purposes (e.g. there is a particular philosophical interest in what the symbol represents). For example, since wffs of the form $\phi \uparrow \psi$ express the same truth function as $v\neg(\phi \wedge \psi)$, adding the operator $\uparrow$ does not substantively extend the system of propositional logic.

A **deviation** from classic logic involves an alteration of the logical system at the **semantic** level (and this has implications for the proof system). In other words, when a logical system is a deviation from classical logic, it will look the same as classical logic in that it will make use of the same symbols and syntax. Where the difference occurs is at the semantic level in terms of how propositional letters are interpreted or how valuation rules for truth-functional operators work. For example, in a three-valued system of logic, the set of wffs in the system will be the same as that of classical logic. However, since a three-valued system of logic denies the principle of bivalence (the principle that every wff is T or F, not both and not neither) and instead asserts that propositional letters are interpreted as T, F, or I (indeterminate), this system of logic deviates from classical propositional logic. Logics that are deviations form classical logic are sometimes referred to as *Deviant Logics*.

Third, a **syntactic variation** of classical logic offers a semantically equivalent but **syntactically** distinct version of classical logic. The systems of classical logic are often characterized in terms of a set of operators, typically a unary operator ($\neg$) and some combination of binary operators (e.g. $\rightarrow, \vee, \wedge, \leftrightarrow$). One example might be a language consisting of $\neg, \vee, \wedge$ and instead of $\rightarrow$, there is the symbol $\mapsto$. Instead of wffs like $\phi \rightarrow \psi$, the $\mapsto$ operator is prefixed to wffs, viz., if ϕ is a wff and ψ is a wff, then $\mapsto \phi\psi$ is a wff in this new system. Semantically speaking, there is no difference between classical logic with its use of $\rightarrow$ and our new syntactically distinct system and its use of $\mapsto$. That is, $\phi \rightarrow \psi$ expresses the same truth function as $\mapsto \phi\psi$, and they both can be used as translations of sentences of the form "if P, then Q". In short, syntactic variations are grammatical alternatives to classical logic while doing same semantic work as their classical logic counterparts.[1]

1.2 Predicate Logic Semantics with Variable Assignments

Before considering various extensions to the language of predicate logic, it is necessary to return to predicate logic semantics. Revising our semantic system slightly will allow for a more general approach to the extensions we will undertake later on.

Recall the following valuation rules for predicate logic (let $\alpha_1, \ldots, \alpha_n$ be any series of names (not necessarily distinct), P be any n-place predicate, and ϕ, ψ are wffs in RL):

[1] The above discussion was primarily drawn from L.T.F. Gamut *Logic, Language, and Meaning*, Vol. 1, pp.156-157.

DEFINITION – RL-VALUATION USING NAMES

1. if $P\alpha_1 \ldots \alpha_n$ is an atomic wff in RL, then $v_{\mathcal{M}}(P\alpha_1 \ldots \alpha_n) = T$ iff $\langle \mathscr{I}(\alpha_1), \ldots, \mathscr{I}(\alpha_n) \rangle \in \mathscr{I}(P)$, otherwise $v_{\mathcal{M}}(P\alpha_1 \ldots \alpha_n) = F$
2. $v_{\mathcal{M}}(\neg(\phi)) = T$ iff $v_{\mathcal{M}}(\phi) = F$
3. $v_{\mathcal{M}}(\phi \wedge \psi) = T$ iff $v_{\mathcal{M}}(\phi) = T$ and $v_{\mathcal{M}}(\psi) = T$
4. $v_{\mathcal{M}}(\phi \vee \psi) = T$ iff $v_{\mathcal{M}}(\phi) = T$ or $v_{\mathcal{M}}(\psi) = T$
5. $v_{\mathcal{M}}(\phi \rightarrow \psi) = T$ iff $v_{\mathcal{M}}(\phi) = F$ or $v_{\mathcal{M}}(\psi) = T$
6. $v_{\mathcal{M}}(\forall x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x) = T$ for every name α in RL.
7. $v_{\mathcal{M}}(\exists x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x) = T$ for at least one name a in RL.

This set of rules makes two simplifications. First, the domain of the valuation function was limited to closed RL-wffs. This simplification left the truth value of open RL-wffs (wffs with free variables, e.g. Px) undefined. That is, our system of logic has a syntax that produces formulas that are wffs but the semantic system does not state whether these wffs are true or false. Second, it assumes that there is an RL-name for every item in the domain of discourse. This assumption allows for specifying the truth value of existentially and universally quantified wffs in terms of non-quantified wffs. For example, $v_{\mathcal{M}}(\forall x)\phi = T$ iff $v_{\mathcal{M}}\phi(\alpha/x)$ for every name a in RL. The problem with this simplification is that it assumes that there is a name for every item in the domain. This assumption is not problematic in the sense that RL lacks enough names to refer to every object for RL has an infinite number of names at its disposal. The problem, instead, is that even with an infinite number of names, there is no guarantee that each item in the domain is named. And if this is the case, then $v(\forall x)Px = T$ even though some unnamed item $u_1 \in \mathcal{D}$ is not in the interpretation of P.

In order to avoid these issues, we reformulate the valuation rule for predicate logic. This reformulation, however, requires some additional notation and complexity.

First, let's begin with the notion of a variable assignment. A variable assignment is a function that has all of the variables in **RL** as its domain (takes the variables as input) and has the items in the domain of the model ($\mathcal{D}$) as its range (takes items in the domain of the model as output).

DEFINITION – VARIABLE ASSIGNMENT

A variable assignment g for a model $\mathcal{M}$ ($\langle \mathcal{D}, \mathscr{I} \rangle$) is a function that assigns to each variable some object in $\mathcal{D}$.

Intuitively, the notion of a variable assignment takes the infinite number of variables from **RL** and assigns each an object in the domain. Thus, for a $\mathcal{D}$ consisting of a single object u, each variable would be assigned to (or refer to) u. For a $\mathcal{D}$ consisting of two objects u_1, u_2, one variable assignment g_1 might assign x and y to u_1, z and the remaining variables to u_2, while another variable assignment g_2 might assign x to u_1 while assigning y, z and the remaining variables to u_2 (since our language involves an infinite number of variables, a variable assignment would assign each variable to either u_1).

We need a way to specify variable assignments so that it is clear which item in the domain is assigned to which variable in the language. Let "$g(x)$" stand for a variable assignment of x. Thus, if $g(x) = u_1$, then $g(x)$ is an assignment of x to u_1.

The next step is to relativize the valuation function not merely to a model ($\mathcal{M}$) but also to a variable assignment (g). That is, we specify the truth value of RL-wffs relative to a model and a way of assigning variables to items in the domain. This valuation is now symbolized as $v_{\mathcal{M},g}$. This can be read as "the valuation relative to a model $\mathcal{M}$ and variable assignment g". This relativization allows us to formulate two different rules for atomic wffs in RL (let α be a name and x be a variable):

> 1a if $P\alpha_1 \ldots \alpha_n$ is an atomic wff in RL, then $v_{\mathcal{M},g}(P\alpha_1 \ldots \alpha_n) = T$ iff $\langle \mathscr{I}(\alpha_1), \ldots, \mathscr{I}(\alpha_n)\rangle \in \mathscr{I}(P)$
> 1b if $Px_1 \ldots x_n$ is an atomic wff in RL, then $v_{\mathcal{M},g}(Px_1 \ldots x_n) = T$ iff $\langle g(x_1), \ldots, g(x_n)\rangle \in \mathscr{I}(P)$

One immediate benefit of this formulation is we now can define the truth value of wffs that have free variables. For example, take the wff Ixx where I is the two-place predicate "x is identical to y". Here we say that $v_{\mathcal{M},g}(Ixx) = T$ iff $\langle g(x), g(x)\rangle \in \mathscr{I}(I)$.

There is, however, a problem with the above valuation rules. The problem is that the valuation rules apply only to atomic wffs containing either names or variables but not to wffs containing both names and variables, e.g. $Lax, Pxyb, Maby$.

To solve this problem, first let's define an RL-term as any name or variable in RL.

> DEFINITION – RL-TERM
>
> An RL-term t is any name or variable in RL.

Second, let's define the notion of a "denotation of a term".

> DEFINITION – DENOTATION OF A TERM
>
> Let $\mathcal{M}$ be a model, g be a variable assignment, t be a term (name or variable). The denotation of t relative to a model and a variable assignment (that is, $[t]_{\mathcal{M},g}$) is:
>
> 1. $\mathscr{I}(t)$ if t is an RL-name, or
> 2. $g(t)$ if t is an RL-variable.

The expression $[t]_{\mathcal{M},g}$ reads the denotation of t relative to a model $\mathcal{M}$ and a variable assignment g.

The denotation of a term t refers to one of two things, depending upon whether t is a name or variable. The denotation refers to the interpretation of t if t is a name or it refers to the variable assignment of t if t is a variable. For example, the denotation of the name a refers to whatever object the name a refers to in the domain (e.g. $\mathscr{I}(a) = 1$), while the denotation of x refers to whatever object the variable assignment of x refers to in the domain (e.g. $g(x) = 2$).

We can now combine the two valuation functions into a single valuation rule that makes us of the notion of a denotation of a term.

> if t is a term and $Pt_1 \ldots t_n$ is an atomic wff in RL, then $v_{\mathcal{M},g}(Pt_1 \ldots t_n) = T$ iff $\langle [t_1]_{\mathcal{M},d}, \ldots, [t_n]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$

To illustrate (here we drop out notation involving relativizing to the model and the variable assignment), take Lax, an atomic wff containing the name a and variable x. We might read this as "Al loves x". Applying our valuation function to this wff: $v(Lax) = T$ iff $\langle [a], [x] \rangle \in \mathscr{I}(L)$. That is, $v(Lax) = T$ iff the ordered pair consisting of the denotation of the name a and the denotation of the variable x are in the interpretation of L. That is, of course, the same as saying that $v(Lax) = T$ iff $\langle \mathscr{I}(a), g(x) \rangle \in \mathscr{I}(L)$. It is important to note that our notion of truth is doubly relativized in that a wff is true relative to a model and relative to a variable assignment. And so, "Al loves x" is true provided, relative to a model and relative to a variable assignment, the ordered pair $\langle Al, [x] \rangle$ is in the interpretation of the two-place predicate Lxy (x loves y).

This modification of the valuation function works well for atomic wffs, but there still remains a problem for existentially and universally quantified wffs.

Consider the wff $(\exists x)Px$ or someone is a person. At first glance, one idea is to say that $v_{\mathcal{M},g}(\exists x)Px = T$ iff $v_{\mathcal{M},g}(Px) = T$. We have a procedure for determining the truth value of such a wff; namely, $v_{\mathcal{M},g}(Px) = T$ iff $\langle [x]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$. Thus, $v_{\mathcal{M},g}(\exists x)Px = T$ iff $\langle [x]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$. This approach is attractive at first glance since it falls in line with our general approach of treating the truth value of complex wffs as compositionally determined. That is to say, similar to how the truth value of $P \wedge Q$ is determined by the truth value of P, Q and the logical operator for conjunction, we now can treat the truth value of $(\exists x)Px$ as being determined by the truth value of Px and the existential quantifier.

But this does not get us the right result since the variable assignment g assigns a specific item from the domain to each variable. This means that x refers to a single item in the domain much like a name. This is problematic because intutiively, the existential quantified wff is true not so long as the denotation of x is in P, but so long as *at least one* item from the domain is in the interpretation of P. In other words, the variable assignment g is just one way that variables may be assigned items in the domain and the existential quantifier requires us to consider variable assignments other than g. What we need then is a way not only to consider other variable assignments. To achieve this, let's introduce the notion of a **variant variable assignment**:

> DEFINITION – VARIANT VARIABLE ASSIGNMENT
>
> Let α be a variable and u be an item in the domain $u \in \mathcal{D}$ of a model, a variant variable assignment g^{α}_{u} is a variable assignment g for a model $\mathcal{M}$ except that it assigns u to α.

In terms of reading a variant variable assignment, where g^{α}_{u} is read as a variable assignment just like g except that the variable α is assigned u. To illustrate, let's consider two examples.

Example 1: Illustration of a variant variable assignment

Suppose there is a variable assignment g where $g(x) = u_1, g(y) = u_2, g(z) = u_3$. The variant variable assignment $g^y_{u_1}$ is just like g except that where $g(y) = u_2$ (that is the variable y is assigned the item u_2) in $g^y_{u_1}$ the variable y is assigned the item u_1. That is, $g^y_{u_1}(y) = u_1$. More explicitly:

- $g = g(x) = u_1, g(y) = u_2, g(z) = u_3$
- $g^y_{u_1} = g^y_{u_1}(x) = u_1, g^y_{u_1}(y) = u_1, g^y_{u_1}(z) = u_3$

Notice that the only difference between g and $g^y_{u_1}$ is that $g^y_{u_1}$ assigns the variable y to u_1 instead of u_2.

It is important to note that a variable assignment and a variant variable assignment might be identical. To see this, consider a second example of a variant variable assignment.

Example 2: Illustration of a variant variable assignment

Suppose there is a variable assignment g where $g(x) = u_1, g(y) = u_2, g(z) = u_3$. Now consider the variant variable assignment $g^x_{u_1}$:

- $g = g(x) = u_1, g(y) = u_2, g(z) = u_3$
- $g^x_{u_1} = g^x_{u_1}(x) = u_1, g^x_{u_1}(y) = u_2, g^x_{u_1}(z) = u_3$

Notice that there is no difference between the variable assignment g and the variant variable assignment $g^x_{u_1}$.

A variant variable assignment provides a way of referring to other variable assignments relative to a model and a variable assignment. That is, given a fixed model and a variable assignment, the variant variable assignment allows for considering other ways that variables might be assigned items in the domain. With the notion of a variant variable assignment in place, it is now possible to formulate the valuation function for universally and existentially quantified wffs.

Definition – RL-valuation

An RL-valuation — for a model $\mathcal{M}$ and variable assignment g — is a function that assigns to each RL-wff a truth value (T or F, not both, not neither) using the following rules (let P be any n-place predicate, $t_1, \ldots, t_n$ be a series of terms (not necessarily distinct), α be any variable, ϕ, ψ any RL-wff):

1. $v_{\mathcal{M},g}(Pt_1 \ldots t_n) = T$ iff $\langle [t_1]_{\mathcal{M},g}, \ldots, [t_n]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$
2. $v_{\mathcal{M},g}(\neg(\phi)) = T$ iff $v_{\mathcal{M},g}(\phi) = F$
3. $v_{\mathcal{M},g}(\phi \wedge \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = T$ and $v_{\mathcal{M},g}(\psi) = T$
4. $v_{\mathcal{M},g}(\phi \vee \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = T$ or $v_{\mathcal{M},g}(\psi) = T$
5. $v_{\mathcal{M},g}(\phi \to \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = F$ or $v_{\mathcal{M},g}(\psi) = T$
6. $v_{\mathcal{M},g}(\forall\alpha)\phi = T$ iff for every $u \in \mathcal{D}$, $v_{\mathcal{M},g_u^\alpha}(\phi) = T$.
7. $v_{\mathcal{M},g}(\exists\alpha)\phi = T$ iff for at least one $u \in \mathcal{D}$, $v_{\mathcal{M},g_u^\alpha}(\phi) = T$.

Let's consider two examples, one involving an existentially quantified wff and one involving a universally quantified wff.

Example 3:

Take the model $\mathcal{M} = \langle \mathcal{D}, \mathscr{I} \rangle$, where $\mathcal{D} = \{1, 2, 3, 4, 5\}$, $\mathscr{I}(N) = \{1, 2, 3, 4, 5\}$, $\mathscr{I}(O) = \{2, 4\}$, $\mathscr{I}(a) = 1$, $\mathscr{I}(b) = 2$, $\mathscr{I}(c) = 3$, $g(x) = 1, g(y) = 2$, and all other variables are assigned 3.

1. Consider the truth value of $(\exists x)Ox$. Given the above valuation rule, $v_{\mathcal{M},g}(\exists x)Ox = T$ since there is one $u \in \mathcal{D}$ such that $v_{\mathcal{M},g_u^x}(Ox) = T$. That is, while it is not the case that $v_{\mathcal{M},g}(Ox) = T$ since the variable assignment g assigns 1 to x, it is the case that there is a variant variable assignment g_u^x where $(\exists x)Ox$ would come out as true. For instance, consider the variant variable assignment g_2^x, viz., where g assigns the variable x to $2 \in \mathcal{D}$. On this variant variable assignment, $(\exists x)Ox$ is true. So, $v_{\mathcal{M},g_2^x}(Ox) = T$.
2. Consider the truth value of $(\forall x)Nx$. Given the above valuation rule, $v_{\mathcal{M},g}(\forall x)Nx = T$ since for every $u \in \mathcal{D}$, it is the case that that $v_{\mathcal{M},g_u^x}(Nx) = T$. That is, $v_{\mathcal{M},g_1^x}(Nx) = T$, $v_{\mathcal{M},g_2^x}(Nx) = T$, $\ldots$, $v_{\mathcal{M},g_5^x}(Nx) = T$.

Exercise 1-1:

Determine the truth values of the following wffs relative to the following model $\mathcal{M}$ and variable assignment g: $\mathcal{M} = \langle \mathcal{D}, \mathscr{I} \rangle$, where $\mathcal{D} = \{1, 2, 3, 4, 5, 6\}$, $\mathscr{I}(a) = 1, \mathscr{I}(b) = 2$ and for all other names $\alpha, \mathscr{I}(\alpha) = 3$; $\mathscr{I}(N) = \{1, 2, 3, 4, 5, 6\}$, $\mathscr{I}(O) = \{1, 3, 5\}$, $\mathscr{I}(G) = \{\langle 2, 1 \rangle, \langle 3, 2, \rangle\}$, $\mathscr{I}(M) = \{2, 4, 6\}$, $g(x) = 1, g(y) = 2$, and all other variables are assigned 3.

1. Oa
2. Mb
3. Mx
4. Gyx

5. $(\exists x)Ox$
6. $(\exists x)Mx$
7. Nx
8. $(\forall x)Nx$
9. $(\forall x)(Ox \vee Mx)$
10. $(\exists x)(\exists y)Gxy$

1.3 Identity

One common relation between members in the domain of discourse is the identity (equality) relation. While we could represent the identity relation as a two-place predicate I and express that a is identical to b as Iab, it is helpful to use a dedicated symbol to express the proposition that two names refer to the same object (relative to a model). Thus, in addition to our list of RL symbols, we add the $=$ symbol. The symbol $=$ will be treated as a logical constant since it will always be treated as the relation of identity. For the negation of identity, viz., to say that two items in a model are not identical, we use $\neq$.

In adding $=$ to the symbols of RL, we are making a superficial extension to our logical system. Nevertheless, while the addition to the system is superficial, this is not to imply that such an additional has no practical value. As we will see, adding $=$ to the symbols of RL allows for a more perspicuous presentation of various sentences that we have interest in expressing, e.g. sentences about specific quantities.

The "=" symbol is intended to express a specific type of two-place predicate relation, namely that of "numerical identity". What is meant when we say that two individuals a and b are numerically identical is that a and b refer to the exact same object. Numerical identity is different from saying two objects are the same in that they are two of the same kind or that they are the same in that they are extremely similar. That is, we might say that two copies of a book are the same, or two identical twins are the same, or two apples in a grocery store are the same, but here we do not mean numerical identity.

In addition, to adding this symbol to our list of symbols, it is necessary to add a rule to our formation rules for creating wffs. Thus, we can add the following formation rule (rule 2) to our existing set of formation (grammatical) rules.

Definition – RL formation rules with identity

1. an n-place predicate P followed by n terms (names or variables) is a wff in **RL**.
2. if α and β are terms (names or variables), then $\alpha = \beta$ is an **RL**-wff.
3. If ϕ is a wff in **RL**, then $\neg(\phi)$ is a wff in **RL**.[a]
4. If ϕ and ψ are wffs in **RL**, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, and $(\phi \leftrightarrow \psi)$ are wffs in **RL**.
5. If ϕ is a wff in **RL** containing a name a and if $\phi(x/a)$ is what results from substituting the variable x for every occurrence of a in ϕ, then $(\forall x)\phi(x/a)$ and $(\exists x)\phi(x/a)$ are wffs in **RL** (provided that '$\phi(x/a)$' is not a wff).
6. Nothing else is a wff in **RL** except that which can be formed by repeated applications of the above.

[a] We will make use of Conventions for parentheses. Thus $\neg(Pa)$ and $\neg Pa$ are both wffs.

For convenience, we will refer to any wffs formed by rule (2) as **identity wffs** or identity statements. Notice that wffs involving $=$ can contain free variables. That is, from a syntactic point of view, both $a = a$ and $x = x$ are RL-wffs.

Example 4: Consider the following examples of RL-wffs involving the identity symbol

1. $a = b$
2. $b = a$
3. $a \neq a$
4. $x = x$
5. $Pa \wedge b = a$
6. $(\forall x)x = x$
7. $(\exists x)(Px \wedge x \neq a)$
8. $(\exists x)Px \wedge x = a$

Next, we introduce a valuation rule to our semantics. Were we to formulate such a rule only for closed identity statements (wffs without free variables), our rule would look as follows:

where α and β are names, $v_{\mathcal{M}}(\alpha = \beta) = T$ iff $[\alpha]_{\mathcal{M}} = [\beta]_{\mathcal{M}}$, otherwise $v_{\mathcal{M}}(\alpha = \beta) = F$.

In other words, the wff $\alpha = \beta$ is true if and only if the item in the domain denoted by α is the same object denoted by β. Another way of thinking about this is to say that the wff $\alpha = \beta$ is true in a model if and only if relative to that model the interpretation of α is identical to the interpretation of β.

Example 5:

Suppose the following model: $\mathcal{M} = \langle \mathcal{D}, \mathscr{I} \rangle; \mathcal{D} = \{2, 4\}, \mathscr{I}(a) = 2, \mathscr{I}(b) = 4, \mathscr{I}(c) = 2, \mathscr{I}(Px) = \{2, 4\}$. First, consider $Pa \wedge a = c$. Here, a must be in the interpretation of P and a and c refer to the same object. Since $\mathscr{I}(a) = \mathscr{I}(c) = 2$, $v(a = b) = T$; and since $\mathscr{I}(a) \in \mathscr{I}(P)$ (the interpretation of a is in the interpretation of P, $v(Pa) = T$. Therefore, $v(Pa \wedge a = c) = T$.

Exercise 1-2: *Given the following model, determine if the following closed identity wffs are true:* $\mathcal{M} = \langle \mathcal{D}, \mathscr{I} \rangle; \mathcal{D} = \{1, 2, 3, 4\}, \mathscr{I}(a) = 1, \mathscr{I}(b) = 2, \mathscr{I}(c) = 1, \mathscr{I}(d) = 1, \mathscr{I}(Px) = \{2, 4\}$

1. $a = b$
2. $b = a$
3. $b = c$
4. $a \neq b$
5. $(\exists x)x = x$
6. $(\forall y)y = y$
7. $(\exists x)Px \wedge x = b$

However, our valuation rule need not be confined to closed wffs. That is, earlier we formulated our semantics so that valuation rules could apply to open formulas. We saw that in order to generalize our valuation rules, it was necessary to relativize the valuation rule not only to the model but to a variable assignment. Thus, our valuation rule for identity statements can be stated in a more general way.

> where α and β are terms (names or variables), $v_{\mathcal{M},g}(\alpha = \beta) = T$ iff $[\alpha]_{\mathcal{M},g} = [\beta]_{\mathcal{M},g}$, otherwise $v_{\mathcal{M}_g}(\alpha = \beta) = F$.

This says that $\alpha = \beta$ is true if and only if the denotation of α relative to the model and the variable assignment is the identical to the denotation of β relative to the model and the variable assignment.

Thus, we can add the above rule to our existing set of valuation rules:

Definition – RL-valuation rules with identity

An RL-valuation — for a model $\mathcal{M}$ and variable assignment g — is a function that assigns to each RL-wff a truth value (T or F, noth both, not neither) using the following rules (let P be an n-place predicate, t be a term, α be a variable, ϕ, ψ any RL-wff):

1. $v_{\mathcal{M},g}(Pt_1 \ldots t_n) = T$ iff $\langle [t_1]_{\mathcal{M},g}, \ldots, [t_n]_{\mathcal{M},g} \rangle \in \mathscr{I}(P)$
2. where α and β are terms (names or variables), $v_{\mathcal{M},g}(\alpha = \beta) = T$ iff $[\alpha]_{\mathcal{M},g} = [\beta]_{\mathcal{M},g}$, otherwise $v_{\mathcal{M}_g}(\alpha = \beta) = F$
3. $v_{\mathcal{M},g}(\neg(\phi)) = T$ iff $v_{\mathcal{M},g}(\phi) = F$
4. $v_{\mathcal{M},g}(\phi \wedge \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = T$ and $v_{\mathcal{M},g}(\psi) = T$
5. $v_{\mathcal{M},g}(\phi \vee \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = T$ or $v_{\mathcal{M},g}(\psi) = T$
6. $v_{\mathcal{M},g}(\phi \rightarrow \psi) = T$ iff $v_{\mathcal{M},g}(\phi) = F$ or $v_{\mathcal{M},g}(\psi) = T$
7. $v_{\mathcal{M},g}(\forall\alpha)\phi = T$ iff for every $u \in \mathcal{D}$, $v_{\mathcal{M},g^{\alpha}_{u}}(\phi) = T$.
8. $v_{\mathcal{M},g}(\exists\alpha)\phi = T$ iff for at least one $u \in \mathcal{D}$, $v_{\mathcal{M},g^{\alpha}_{u}}(\phi) = T$.

Exercise 1-3: *Given the following model and variable assignment, determine if the following closed identity wffs are true:* $\mathcal{M} = \langle \mathcal{D}, \mathscr{I} \rangle; \mathcal{D} = \{1, 2, 3, 4\}, g(x) = 1, g(y) =$

$2, \mathscr{I}(a) = 1, \mathscr{I}(b) = 2, \mathscr{I}(c) = 1, \mathscr{I}(d) = 1, \mathscr{I}(Px) = \{2, 4\}$

1. $x = x$
2. $x = y$
3. $a = x$
4. $a \neq x$
5. $(\exists x)Px \wedge x = b$

Earlier it was noted that adding $= \neq$ to RL amounts to a superficial extension but that it nevertheless has value. One reason this extension is valuable is becaues it allows for a more readable translation of various English expressions into the language of predicate logic. In what follows we will use the identity symbol to translate three different types of formulas:

1. Numerical propositions:
   - There is at least n Ps where n is some number and P is a predicate
   - There is at most n Ps where n is some number and P is a predicate
   - There is exactly n P where n is some number and P is a predicate
2. Sentences with exceptive phrases: Every a is P **except** b where a and b are terms and P is a predicate
3. Superlatives: a is the **best** P, where a is a name and P is a predicate

1.3.1 Numerical propositions

Let's begin with numerical propositions, e.g. propositions of the form, "at least n are P", "at most n are P", and "there are exactly n are P", where n is some numeral and P is some property. Previous sections have shown how it is possible to translate "all ns are P" and "some n is P" with the universal and existential quantifier, respectively. In this section, propositions involving *specific quantities* are expressed by combining the identity relation along with the existential and universal quantifier.

Consider the following propositions:

1. There are at least two zombies
2. There are at least three zombies
3. There is at most one zombie.
4. There are at most two zombies
5. There is exactly one zombie
6. There are exactly two zombies

First, let's begin with propositions of the form **at least n Ps**. Proposition (1) should not be translated as $(\exists x)(\exists y)(Zx \wedge Zy)$ since the object referred to by each of the bound variables could be the same object. That is, the existence of a single object that is Z would make $(\exists x)(\exists y)(Zx \wedge Zy)$ true. Thus, to ensure that there are *at least two items that are* Z, it is necessary to ensure that the objects referred to by the bound variable are *not* identical. Proposition (1) can thus be translated as follows:

1. $(\exists x)(\exists y)[(Zx \wedge Zy) \wedge (x \neq y)]$

This wff says that there is a zombie x and a zombie y, and the zombie x is not the same thing as zombie y. Thus, there are at least two distinct zombies.

Relying on the strategy we used to translate proposition (1), we can translate proposition (2) by adding another quantifier and two more subformulas involving non-identity:

2. $(\exists x)(\exists y)(\exists z)[(Zx \wedge Zy \wedge Zz \wedge (x \neq y) \wedge (x \neq z) \wedge (y \neq z)]$

This strategy can be extended for any expression of the form "there are at least n Ps":

At least one zombie	$(\exists x)Zx$
At least two zombies	$(\exists x)(\exists y)[(Zx \wedge Zy) \wedge (x \neq y)]$
At least three zombies	$(\exists x)(\exists y)(\exists z)[(Zx \wedge Zy \wedge Zz \wedge (x \neq y) \wedge (x \neq z) \wedge (y \neq z)]$
At least n zombies	$(\exists x_1 \ldots \exists x_n)(Zx_1 \wedge \ldots \wedge Zx_n \wedge x_1 \neq x_2 ... x_{n-1} \neq x_n)$

Second, let's turn to expressions that say there are at **at most n Ps**. One thing to note is that this expression does not assert that there are any Ps. For example, if Tek utters (3), he does not assert that there are any zombies but instead says that if there are any zombies, there are no more than two zombies.

One strategy for translating propositions of the form **at most n Ps** is to negate an **at least n Ps** proposition. For example, suppose we wanted to translate "there are at most 2 Ps". One way of expressing this is by saying it is not the case that there are at least three Ps. To see this in practice, consider proposition (3), which says that there is at most one zombie. We translate (3) by negating the proposition "there are at least two zombies." This translation is $\neg(\exists x)(\exists y)[(Zx \wedge Zy) \wedge (x \neq y)]$. Using some equivalence rules, we can transform the negated existential wff into the following:

3. $(\forall x)(\forall y)((Zx \wedge Zy) \rightarrow (x = y))$

This wff says that for every x and every y, if x and y are Z, then x is identical to y. That is, if there is a zombie (call it "Tek"), then every other object that is identical to Tek, which is to say that if there is a zombie, then Tek is the only one.

Using the same strategy as above, we can translate proposition (4) first as $\neg(\exists x)(\exists y)(\exists z)[(Zx \wedge Zy \wedge Zz \wedge (x \neq y) \wedge (x \neq z) \wedge (y \neq z)]$ and then using equivalence rules, we can translate it ultimately as:

4. $(\forall x)(\forall y)(\forall z)((Zx \wedge Zy \wedge Zz) \rightarrow (x = y \vee x = z \vee y = z))$

This strategy can be extended for any expression of the form "there are at least n Ps":

At most one zombie	$(\forall x)(\forall y)((Zx \wedge Zy) \rightarrow (x = y))$
At most two zombies	$(\forall x)(\forall y)(\forall z)((Zx \wedge Zy \wedge Zz) \rightarrow (x = y \vee x = z \vee y = z)]$
At most n zombies	$(\forall x_1) \ldots (\forall x_n)((Zx_1 \ldots Zx_n) \rightarrow$ $(x_1 = x_2 \vee x_1 = x_3 \ldots \vee x_1 = x_n$ $\vee x_2 = x_3 \ldots x_2 = x_n$ $\vee x_{n-1} = x_n))$

Let's finish with expressions of the form **there is exactly n Ps**. Another way of expressing **there is exactly n Ps** is as **there is at least n Ps and at most n Ps**. For consider that were we to say that there are at least five apples and at most five apples, we would be saying that there are exactly five apples. Thus, proposition (5) that "there are exactly one zombie" expresses the proposition "there at least one zombie and there is at most one zombie". Since we already have strategies for translating there is at least n Ps and there are at most n Ps, we only need to conjoin these strategies to translate **there is exactly n Ps**.

Applying this combination of strategies to (5), "there is exactly one zombie" can be translated as follows: $(\exists x)Zx \wedge (\forall x)(\forall y)((Zx \wedge Zy) \rightarrow (x = y))$. This says that there exists a zombie and that every zombie is identical to the one that exists. More compactly, (5) can be translated as follows:

5. $(\exists x)(\forall y)((Zx \wedge Zy) \rightarrow x = y)$

Finally, consider (6), which says that there are exactly two zombies. Here we want to say that there exists an x and exists a y that are Z such that $x \neq y$ (this will ensure we have two zombies) and that for every z that is a zombie Z, z will be identical to either x or y (this will ensure that there are no more than two zombies. Thus, (6) can be translated as follows:

6. $(\exists x)(\exists y)(\forall z)((Zx \wedge Zy \wedge x \neq y) \wedge (Zz \rightarrow z = x \vee z = y)$

This strategy can be extended for any expression of the form "there is exactly n Ps":

Exactly one zombie	$(\exists x)(\forall y)((Zx \wedge Zy) \rightarrow x = y)$
Exactly two zombies	$(\exists x)(\exists y)(\forall z)((Zx \wedge Zy \wedge x \neq y) \wedge (Zz \rightarrow z = x \vee z = y)$
Exactly n zombies	$(\exists x_1)\ldots(\exists x_n)(\forall y)((Zx_1 \wedge Zx_2 \wedge, \ldots, \wedge Zx_n$ $\wedge x_1 \neq x_2, \wedge \ldots, \wedge x_{n-1} \neq x_n)$ $\wedge(Zy \rightarrow y = x_1 \vee y = x_2, \vee \ldots \vee y = x_n))$

Exercise 1-4: *Translate the following English sentences into the language of predicate logic.*

1. There is at least one pumpkin
2. There are no pumpkins
3. There are at least three pumpkins
4. There are at most three pumpkins
5. There are exactly three pumpkins

1.3.2 Exceptive phrases

The identity symbol can be used to translate sentences with exceptive phrases. Let's take a sentence with an exceptive phrase as a closed sentence of the following form: **Every a is P except b** (where a and b are names and P is a predicate). Intuitively, sentences with exceptive phrases attribute a property P to a collection of things A but also specify that a thing b (that is a member of A) or subset B of A lacks P.

While our focus will be on exceptive phrases of the form **Every a is P except b**, there are a number of ways to use exceptive phrases in a sentence.

1. All books are good **except** *War and Peace*
2. Every book is good **other than** *War and Peace*

3. All books are good **besides** *War and Peace*
4. Every book **but** *War and Peace* is worth reading

Notice that in each sentence, the property of *goodness* is attributed to everything that is a book, but this property is specified as not belonging to a specific book, namely *War and Peace*. Again, this is just one variety of exceptive phrase for we could specify that every book is good except books that have a certain property, e.g. except romantic fiction.

Consider the sentence "Every book is good except *War and Peace*". First, let's treat this sentence as though it doesn't contain the exceptive phrase. The sentence without the exceptive phrase says "for every x, where x is a book, x is good." This sentence can be translated straightforwardly as follows:

$$(\forall x)(Bx \rightarrow Gx)$$

Next, we need to add the exceptive clause. To do this, we cannot simply conjoin a formula that says that *War and Peace* is not good, e.g. $\neg Gp$ for consider that *War and Peace* is a book, the sentence says all books are good, and then turns around and says that *War and Peace* is not a good book. This would turn the seemingly contingent sentence into a contradiction. What we need to do instead then is to subtract *War and Peace* from the items over which the universal quantifier operates. To do this, we can specify that the items over which the quantifier operators should *not* be identical to the item (or items) we wish to exclude. That is, we wish to say that every book that is not identical to *War and Peace* is good. In other words, we wish to say that for all x if x is a book B and x is not identical to *War and Peace*, then x is good G:

$$(\forall x)((Bx \wedge x \neq p) \rightarrow Gx))$$

What about our earlier sentence that excluded not merely a single item but a class of items? That is, how would we translate the sentence "all books are good except works of romantic fiction". Here we only need to make a small

$$(\forall x)(\forall y)((Bx \wedge Ry \wedge x \neq y) \rightarrow Gx)$$

This says that for every book B that is not romantic fiction R, that book is good G. That is, every book is good except works of romantic fiction.

Exercise 1-5: *Translate the following English sentences into the language of predicate logic.*

1. Every movie is good except *The Terminator*
2. All of my friends are nice except Tek.
3. Every candidate is corrupt except Trump.
4. All movies are great except horror movies.
5. All movies are great except horror movies and *The Terminator*.

1.3.3 Superlatives

Finally, the identity symbol is useful for translating superlative sentences. These are sentences of the form **a is the best P**, where a is a name and P is a predicate. To translate sentences of this variety, we will make use of a two-place predicate Gxy or "x is greater than y". Consider the following sentences:

1. a is greater than b
2. a is the greatest
3. a is the worst

There is already a method for translating sentences like (1). We simply use the two-place predicate G and insert the names in their respective places:

1. Gab

Next, to translate (2), we cannot simply use the formula $(\forall x)Gax$ for this says that a is greater than every object (which would include a). When one says that a particular object a is the greatest, they are not also saying that a is greater than itself. Rather, they are saying that a is greater than every other thing that is not identical to a. In other words, $(\forall x)Gax$ would imply Gaa which states that a is greater than itself. What we wish to express then is that a is greater than everything that is not identical to a. This can be achieved with the following formula:

2. $(\forall x)(x \neq a \rightarrow Gax)$

Here the formula states that for every x that is not identical to a, a is greater than that item.

Finally, consider (3), which says that a is the worst. Here again, we wish to say that every other object that is not identical to a is greater than a. This is achieved by the following formula:

3. $(\forall x)(x \neq a \rightarrow Gxa)$

In contrast to (2), (3) says that every individual x that is not a is greater than a.

Exercise 1-6: *Translate the following English sentences into the language of predicate logic.*

1. A man smiled and another frowned
2. If Tek ate exactly one cookie, then there are exactly two cookies.
3. All politicians are crooks except Bernie Sanders
4. Tek loves everyone except Sally
5. *Grouplove* is the best music group.
6. Donald Trump is the best president.
7. Donald Trump is the worst president.

1.3.4 Derivation rules for identity

Two derivation rules are formulated for wffs involving =. The first is identity introduction. This rule states that at any point in the proof, for any name a, you may write the wff $a = a$.

Definition – Identity Introduction $= I$

At any point in the proof, you may write $a = a$ for any name a.

$\vdash a = a$

The intuition behind this rule is that every object is identical to itself.

For example, consider the following derivation $\vdash (\forall x)(x = x)$.

1	$\neg(\forall x)(x = x)$	A
2	$a = a$	$= I$
3	$(\forall x)(x = x)$	$\forall I$, 2
4	$(\forall x)(x = x)$	$\neg I$, 1–3

The second derivation rule for wffs involving the identity symbol is called *identity elimination*. This rule states that for any names a and b, from $a = b$ and a wff ϕ where ϕ contains the name a, you may derive $\phi(b/a)$. That is, you may substitute b for a in the wff ϕ.

Definition – Identity Elimination $= E$

From $a = b$ and a wff ϕ, you may substitute a for b in ϕ or b for a in ϕ.

$a = b, \phi \vdash \phi(a/b)$ or $a = b, \phi \vdash \phi(b/a)$

The intuition behind this rule is that if the names a and b refer to the same thing, then whatever is true of a is true of b, and vice versa.

For example, consider the following proof:

1	$a = b$	P
2	Lac	P
3	Lbc	$= E$, 1, 2

Next, consider the following derivation of Lca from $a = b$ and Lcb.

1	$a = b$	P
2	Lcb	P
3	Lca	$= E$, 1, 2

In this proof, since $a = b$ and it is the case that Lcb, b can be replaced with a in Lcb.

Exercise 1-7: *Provide a derivation for the following entailments*

1. $a = b, (\forall x)Lxa \vdash (\forall x)Lxb$
2. $a = b, b = c, (\forall x)Px \rightarrow Qc \vdash Qb$
3. $Pa \vdash (\exists x)(Px \wedge x = a)$
4. Prove that identity is transitive, i.e., $\vdash (\forall x)(\forall y)(\forall z)(x = y \wedge y = z \rightarrow x = z)$.

This states that for any three items, if the first is identical to the second, and the second is identical to the third, then the first is identical to the third.

5. Prove that identity is symmetric, i.e., $\vdash (\forall x)(\forall y)(x = y \to y = x)$. This states that for every item, if the first is identical to the second, then the second is identical to the first.
6. Prove that identity is reflexive, i.e., $\vdash (\forall x)(x = x \to x = x)$. This states that everything item is identical to itself.

1.4 Functions

Consider the following sentences:

1. Jon is Irish.
2. Tek's father is Irish.

The translation of (1) is straightforward. Where Ix stands for the one-place predicate "x is Irish" and j stands for the name "Jon", (1) can be expressed as Ij. What about the translation of (2)? At first glance, our translation of (2) might be guided by our translation of (1). Just like the name "Jon" picks out an individual *Jon*, "Tek's father" purportedly picks out a single entity, namely the specific individual that is the father of Tek. Thus, it seems plausible to say that "Tek's father" is a proper name and to treat as such in **RL**. Under this thought process, (2) would be translated as It.

But this translation overlooks the internal semantics that the possessive "Tek's father" has that a name like "Tek", "Jon", or "Barack Obama" lacks. A name like "Jon" means what it does simply in virtue of the name alone, while the expression "Tek's father" means what it does in virtue of two semantic ingredients: (i) the meaning of the name "Tek" and (ii) the meaning of "being a father of x". In short, treating the possessive "Tek's father" as though it were a name in **RL** would ignore semantically significant parts of the expression.

Furthermore, we reason differently with names as opposed to possessives like the one found in (2). In the case of (2), we can reason from "Tek's father is Irish" to "Someone's father is Irish". If the possessive was simply a name, then it seems we would be limited to reasoning to "someone is Irish" rather than "someone's father is Irish".

Another thought is to treat possessives like "Tek's father" not as names bupt as predicates combined with names. However, with n-place English predicates, the addition of n names in their correct places yields a sentence that expresses a proposition. That is, if an n-place predicate takes n-names, it yields a wff that takes a value of true or false. Thus, in the case of the one-place predicate "x is happy", the substitution of a name "Barack Obama" for x results in a sentence that expresses a proposition: *Barack Obama is happy.* Much like predicates, possessives can also be thought of as having places, but an n-place possessive yields not a proposition that takes a value of true or false but instead takes a value of an object in the domain. For example, replacing "Barack Obama" for x in the one-place possessive "x's book" results in "Barack Obama's book", a phrase that refers to a single item rather than a truth value.

To fit expressions like possessives into our logical language, in this section we add symbols to express functions. What is a function? Suppose A and B are sets (not necessarily distinct). A function f is a relation of each member from A to exactly one member from B, where A and B are not necessarily distinct. In a function f,

1. members from A are called the inputs (or arguments) of the function, while the members from B that are put in relation to members from A are called the outputs (or values) of the function
2. the set A itself is often called the *Domain of the function*, while the set B itself is called the *Codomain of the function*
3. when each and every member from A is related to exactly one member in B, we say that the function *covers* A
4. while each and every member from A is related to exactly one member in B, it is not the case that every member from B is in a relation with a member from A. That is, some members from B might not be in a relation at all.
5. Sometimes, a function f is said to be defined *on* a set. Were we to add sets A and B together, the result of the set is what is called the *union of A and B*.

1.4.1 Symbols and syntax

Symbolically, n-place function symbols are introduced using f with or without positive integer subscripts to ensure there are an infinite number of function symbols and superscripted positive integers to indicate whether the function is a 1-place, 2-place, or n-place function.

Example 1: Use of the function symbol

1. f^1 - one-place function
2. f_3^2 - two-place function
3. f_3^5 - five-place function

Generally, however, superscript integers are not used as the number of places of the function will be clear from the rest of the notation involving the function.

In terms of syntax, there is no change to the formation rules themselves. What is instead expanded upon is the definition of an RL-term. To achieve this, let's define the notion of a functional term

Definition – functional term

If f^n is an n-place function symbol (where $n > 0$) and $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a series of n terms (not necessarily distinct), then $f^n(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a functional term.

Note that functional terms have a certain number of places (that is to say, it takes a certain number of inputs/arguments). The inputs of the function are written to the right of the function symbol within a pair of parentheses and each input is separated by commas.

We read $f(a)$ as "the f of a" and $f_2(a, b)$ as "the f_2 of a and b".

Example 2: Consider the following functional terms

1. $f(a)$ - the father of Al
2. $f(b)$ - the brother of Bethany
3. $f^2(b, a)$ - the sum of b and a
4. $f(f(a))$ - the father of the father of Al
5. $f(f(a, b), b)$ - the sum of the sum of a and b and b

One thing to note about functional terms is that unlike names and variables, functional terms are *complex*. In the case of names and variables, there is no internal syntactic structure whereas functional terms consists of a function symbol and at least one other term.

Given that functional terms are composed of a function symbol and a term and that functional terms are themselves terms, it follows that the term(s) of a functional term can itself be a functional term. For example, consider that $f_1(f_2(a), a)$ is a functional term yet one of the terms of f_1 is a functional term. More concretely, suppose that f stands for the two-place function "the sum of x and y". A functional term like $f(5, 3)$ would express "the sum of 5 and 3". But suppose we were to write $f(f(3, 2), 1)$. Here we would be expressing "the sum of the sum of 3 and 2, and 1". Still further $f(f(3, 2), f(2, 1)$, "the sum of the sum of 3 and 2 and the sum of 2 and 1." A less mathematical example might be the one-place function f (the father of x). Using this function, we might write $f(f(a))$, which states "the father of the father of a".

Having added functional terms to the class of terms, it is worthwhile to reformulate our definition of an RL-term:

Definition – **RL** term

An RL-term is any name, variable, or function term in RL.

The expanded notion of a term that includes function symbols does not require any revision to the formation rules of **RL**:

If P is an n-place predicate and if $t_1, t_2, \ldots t_n$ are a series of terms (not necessarily distinct), then $Pt_1, \alpha_2, \ldots t_n$ is a wff.

Let's consider two examples of RL-wffs involving functional terms.

Example 3: wff with functional term

Suppose P is a one-place predicate and f is a function symbol. Thus, P followed by one term is a wff and so $Pf(\alpha_1)$ is an **RL**-wff. Assuming P stands for "is old" and $f(x)$ the expression x's book, and α_1 is a name referring to Barack Obama, then $Pf(\alpha_1)$ can be translated as "Barack Obama's book is old".

Example 4: wff with functional term

Suppose W is a two-place predicate, f is a function symbol, and d is a name. Thus, $(\forall x)Wxf(d)$ is an **RL**-wff. Assuming W stands for "x wants y" and $f(x)$ the expression x's money, and d refers to Donald Trump, then $(\forall x)Wxf(d)$ can be translated as "Everyone wants Donald Trump's money".

1.4.2 Semantics

Next, we turn to the semantics of predicate logic involving function symbols. First, we have an interpretation of function symbols. To define the notion of a function in precise terms, we need the notion of a Cartesian Product.

DEFINITION – CARTESIAN PRODUCT

A Cartesian Product is a mathematical operation that returns a set form at least two (not necessarily distinct) sets such that that Cartesian Product of the sets A and B (represented as $A \times B$) is the set consisting of all the ordered pairs of $\langle a, b \rangle$ where $a \in A$ and $b \in B$.

Example 1: Cartesian Product of A and B

Suppose $a, b \in A$ and $1, 2 \in B$. The Cartesian Product of A and B ($A \times B$) is $\{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle\}$.

Example 2: Cartesian Product of B and C

Suppose $1, 2 \in B$ and $a, b, c \in B$. The Cartesian Product of B and C ($B \times C$) is $\{\langle 1, a\rangle, \langle 1, b\rangle, \langle 1, c\rangle, \langle 2, a\rangle, \langle 2, b\rangle, \langle 2, c\rangle\}$.

With the notion of a Cartesian Product in place, we can define a function (f) as a subset of a Cartesian Product.

Definition – function

a function f from sets X to Y is a subset of the Cartesian Product of X and Y subject to the following condition: that for every $x \in X$, x is the first component of one and only one of the members of the subset of $X \times Y$.

This definition of a function ensures that every item in X is related to one and only one item in Y.

Let's consider an example. First, suppose again we have $a, b \in A$ and $1, 2 \in B$. We saw the Cartesian Product of A and B ($A \times B$) was $\{\langle a, 1\rangle, \langle a, 2\rangle, \langle b, 1\rangle, \langle b, 2\rangle\}$. Notice that here it is not the case that for every item x in A, x is the first component of one and only one of the members of the subset $A \times B$. For example, note that the element b is the first member of the ordered pair $\langle b, 1\rangle$ and the first member of the ordered pair $\langle b, 2\rangle$. And so the Cartesian Product of $A \times B$ is not a function.

To define a function from A to B, it is necessary that it be a subset of the Cartesian Product from A to B that adheres to the condition that for every item $x \in A$, x is the first component of one and only of of the subset $A \times B$. For example, we might define f_1 as the subset $\{\langle a, 1\rangle, \langle b, 1\rangle\}$, or we could define f_2 as the subset $\{\langle a, 2\rangle, \langle b, 1\rangle\}$. In both of these examples, each item in A is related to one and only one of the items in B.

Definition – interpretation of a function

Relative to a model $\mathcal{M}$, for sets A and B, both of which are subsets of the domain $\mathcal{D}$, the interpretation of a function, $\mathscr{I}(f)$, from A to B is a subset of the Cartesian Product ($A \times B$) such that for every item $x \in A$, x is the first component of one and only one of the members of the subset of $A \times B$.

In other words, the interpretation of an n-place function symbol is simply a mapping of members of the domain $\mathcal{D}$ to members of the domain $\mathcal{D}$. The interpretation of a function can be represented as a set of ordered pairs where the first item in the pair is the input of the pair and the second item is the output.

In the case of functional terms, the interpretation of a functional term is the item in $\mathcal{D}$ that the function maps an item in $\mathcal{D}$ to. That is, it is the output of the function given the function's input.

Example 3:

Consider the function $f = \{\langle 1, 2\rangle, \langle 2, 3\rangle, \langle 3, 1\rangle\}$, $\mathscr{I}(a) = 1, \mathscr{I}(b) = 2, \mathscr{I}(c) = 3$, the interpretation of the functional term $f(a)$ is 1. That is, $\mathscr{I}(f(a)) = 1$. Furthermore, $\mathscr{I}(f(b)) = 3$, $\mathscr{I}(f(c)) = 1$.

Example 4:

Consider the function $f = \{\langle Tek, Liz\rangle, \langle Liz, Jon\rangle, \langle Jon, Liz\rangle\}$, $\mathscr{I}(a) = Tek, \mathscr{I}(b) = Liz, \mathscr{I}(c) = Jon$. Here $\mathscr{I}(f(a)) = Liz$, $\mathscr{I}(f(b)) = Jon$, $\mathscr{I}(f(c)) = Liz$.

Earlier we made use of the notion of a denotation of a term. Since functional terms are terms, it is necessary to expand this notion to include functional terms. In sum, the denotation of a t when t is a functional term is the interpretation of the functional term (as we have illustrated above). That is, $[t]$ when t is $f(a)$ is the interpretation of $f(a)$, and the interpretation of $f(a)$ is the item in $\mathcal{D}$ that f maps a to.

DEFINITION – DENOTATION

For any model $\mathcal{M}$, variable assignment g, and term t, the denotation of t ($[t]_{\mathcal{M}}, g$) is defined as follows:

1. $\mathscr{I}(t)$ if t is a name,
2. $g(t)$ if t is a variable
3. $\mathscr{I}(f)([t_1], \ldots [t_n])$ if t is $f(t_1, \ldots, t_n)$

With the notion of an interpretation in place, we turn to the valuation function. Here we do not make any changes to the specification of the valuation function. Namely, we keep the part of the valuation function that states that for any n-place predicate P and terms $\alpha_1, \ldots \alpha_n$, $v_{\mathcal{M},g}(P\alpha_1 \ldots \alpha_n) = T$ iff $([\alpha_1]_{\mathcal{M}} \ldots [\alpha_n]_{\mathcal{M}} \in \mathscr{I}(P)$.

1.5 Definite descriptions

In English, when we wish to refer to a specific individual, one way that we do this is by using that entity's name. For example, if I wish to refer to *Barack Obama*, I use the proper name "Barack Obama". Similarly, in classical predicate logic, when we wish to refer to a specific individual, we use a name, e.g. b. In English, however, we often refer to specific individuals in other ways besides using the thing's proper name. And, it is useful to have another way of referring to specific individuals since we do not always know the name of the thing that we wish to refer. One way that we refer to specific individuals is by giving a **description** that purports to describe that specific individual alone.

There are at least two ways to describe things, only one of which will be of concern to us here. Indefinite descriptions are phrases of the form "an A" or "a A". For example, "a book" or "a person I once knew". These types of descriptions are thought to describe an object but do so in a way that is ambiguous, viz., the description potentially fits more than one object. And, indefinite descriptions do not require any expansion of RL since expressions like "a man is happy" can be captured using the existential quantifier, e.g. $(\exists x)(Mx \wedge Hx)$.

Definite descriptions are said to be phrases of the form "*the A*". For example, "the man" or "the first president of the United States". These types of descriptions are thought to describe an object and do so in a way that the description fits one and only one object.

This characterization of descriptions is somewhat crude for at least two reasons. First, there are a number of other expressions that might qualify as indefinite or definite descriptions but they don't take the form of "a A" or "the A respectively. Some have argued that proper names can be construed as definite descriptions. For example, "Barack Obama" is shorthand for "the 44th president of the United States" or some other description that would only fit Barack Obama. And at least at first glance, this makes intuitive sense for what is a name besides a shorthand way of referring to the complete set of descriptions one has in one's mind. Another example might involve descriptions that involve possessive pronouns. For example, suppose Tek uttered "My father is going to be angry when he gets the news." The description "my father" seems to fit one and only one object Tek's father but does not have the form of "the father of Tek". A second problem with construing definite descriptions in this way is that there are a number of instances of "an A" or "the A" that don't qualify as indefinite and definite descriptions respectively. For example, if Tek uttered "John is the man" or "Mary is a doctor", the expressions *the man* and *a doctor* are predicates rather than as kinds of descriptions. Let's put aside these issues, however, and treat indefinite and definite descriptions as expressions of the form "a A" and "the A" respectively.

Consider the following definite descriptions:

1. The first woman in space
2. The mother of Barack Obama
3. The present Queen of England

At first glance, we might treat these descriptions like names. However, there are at least two problems with doing this. First, in contrast to names which are simple expressions, definite descriptions are composite expressions. That is, the definite descriptions above consist of (i) the definite article "the" along with (ii) a predicate expression that applies to one individual.

Second, definite descriptions are similar to proper names in that they purportedly pick out one and only one (a unique) individual. That is, insofar as "the first black president of the United States" is a definite description, it purports to pick out a single individual. However, definite descriptions are different from proper names in that they do not pick out a unique individual by simply referring to that thing. That is, we can view proper names as directly referring to their objects, viz., as acting as indices or name-tags of things. In contrast, definite descriptions refer to their objects not directly but providing a description that would only fit a single item. For example, "the first black president of the United States" refers to a unique object by providing a description that fits one and only one object. Like a name it picks out a single individual, but unlike a name it picks out this object by providing a description that fits one and only one object, namely Barack Obama.

1.5.1 Symbols and syntax of definite descriptions

To capture the composite structure (definite article and predicate), we first expand RL by adding a new symbol. That is, let's refer to ι as the "iota" or "the" operator (technically, the symbol used for definite descriptions is the reversed iota symbol).

Next, we need to adjust the syntax of RL by modifying both our definition of a RL-term and a RL-wff. Namely, we need to include the following addition to our definition of a term:

> DEFINITION – IOTA-TERM (DEFINITE DESCRIPTION)
>
> If ϕ is a RL-wff and x is a variable, then $\iota x\phi$ is a iota-term (definite description).

Since we are including a new kind of expression in the class of RL-terms, it is necessary to make this explicit in our official definition of an RL-term.

DEFINITION – RL-TERM

An RL-term t is any name, variable, functional term, of iota-term in RL.

Let's consider some examples of RL-terms involving the iota operator.

Example 1: Definite descriptions

1. $\iota x Mx$ - the man
2. $\iota x(Ax \wedge Mx)$ - the angry man
3. $\iota x(Ax \wedge Mx \wedge (\forall y)Lxy)$ - the angry man that loves everyone
4. $\iota x(Mx \wedge x = \iota y Ky)$ - the man who is identical to the king

Now that we have expanded the notion of an RL-term, we modify our definition of an RL-wff to accommodate the use of terms involving descriptions.

If P is a n-place predicate, where $n \geq 1$, and $t_1 \ldots t_n$ are terms (not necessarily distinct) in RL, then $Pt_1 \ldots t_n$ is a wff in RL.

Example 2: Consider the following RL-wffs involving the iota operator

1. $P(\iota x Mx)$ - The man is a person
2. $L(a, \iota x Mx)$ - Al loves the man
3. $b = \iota x Mx$ - Bob is identical to the man
4. $(\exists x)Wx \wedge = \iota y Dy$ - Some woman is identical to the doctor

1.5.2 Semantics of definite descriptions

Next, we turn to the semantics of definite descriptions. Since $\iota x \phi$ is a term, we specify the denotation of the term $[\iota x \phi]$ as the single object u_1 that $\iota x \phi$ denotes. The idea here is that definite descriptions have meaning in themselves, and the meaning of any definite description is simply the object it denotes (or refers to).

$[\iota x \phi]_{\mathcal{M},g}$ is the unique individual $u \in \mathcal{D}$ such that $v_{\mathcal{M},g^x_u}\phi = T$

The idea behind this treatment of definite descriptions is *not* that the object u_1 that the definite description refers to is the unique object $u_1 \in \mathcal{D}$ that would make the **wff** containing the definite description true. For example, take the sentence "the first black president is a republican". Intuitively, the definite description "the first black president" would refer to *Barack Obama* but even if this is the referent of the definite description, this item would not make the sentence true (since Barack Obama is a democrat rather than a republican).

The idea instead is that the denotation of the definite description is the object that would make "x is the first black president" true, viz., the $u \in \mathcal{D}$ that would make $v(Bx) = T$. That is, $[\iota x Bx]_{\mathcal{M},g}$ is the unique individual $u_1 \in \mathcal{D}$ such that $v_{\mathcal{M},g^x_u}(Bx) = T$.

However, this interpretation of wffs involving the iota operator is problematic for a number of reasons.

1.5.3 Problems with treating definite descriptions as names

The first problem with treating definite descriptions as denoting phrases (like names) is when there is more than one $u \in \mathcal{D}$ that would make $\mathcal{M}, g^x_u\phi$ true, then $[\iota x\phi]_{\mathcal{M},g}$ is undefined. Suppose we were to look out and see two men with hats and then utter "the man in the hat is menacing". Suppose the first man is u_1 and the second man is u_2. It seems that "the man in the hat" is undefined since there is no unique individual that would make $v_{\mathcal{M},g^x_u}Mx = T$ since both u_1 and u_2 could make the wff true. And, if "the man in the hat" is undefined, then so is the sentence "the man in the hat is menacing". And if "the man in the hat is menacing" is undefined, then it is neither true nor false (a violation of the principle of bivalence).

The second problem involves a set of cases where the definite descriptions apparently refer to non-existent objects. That is, there isn't any $u \in \mathcal{D}$ such that $v_{\mathcal{M},g^x_u}\phi = T$. The main problem is that if there is no $u \in \mathcal{D}$ such that $v_{\mathcal{M},g^x_u}\phi = T$ then $\iota x\phi_{\mathcal{M},g}$ is undefined. We will consider two cases of this second problem.

The first case involves the law of the excluded middle (LEM). The LEM states that $\phi \vee \neg(\phi)$ is a tautology (valid wff) for any ϕ. But consider the proposition "The present King of France is either bald or not bald".

Suppose we translate "the present King of the France is bald" as $B(\iota xKx)$. Since there is no unique individual $u \in \mathcal{D}$ such that $v_{\mathcal{M},g}(Kx) = T$, $[\iota xKx]_{\mathcal{M},g}$ is undefined. This has ramifications for $B(\iota xKx)$ because if $[\iota xKx]_{\mathcal{M},g}$ is undefined, then so is $B(\iota xKx)$, and if $B(\iota xKx)$ is undefined, then we have a wff in our language that is neither true nor false. And, if we have a wff that is neither true nor false, then we have a violation of the principle of bivalence.

In short, if we treat definite descriptions that fail to refer as undefined, then we have a counterexample to the law of excluded middle. Namely, $\phi \vee \neg(\phi)$ is not a tautology when either ϕ or $\neg(\phi)$ refers to a non-existent.

Another option might be to treat definite descriptions that fail to refer as false, but this would serve us no better since it would mean that both ϕ and $\neg(\phi)$ are false. And again, $\phi \vee \neg(\phi)$ is not a tautology.

The second case involves negative existentials. Negative existentials are propositions where a non-existent is the subject of a proposition of the following form:

> x does not exist (where x is a non-existent entity)

Examples of negative existentials include propositions like "Cerberus does not exist" and "The unicorn in my bedroom does not exist". Suppose that S is a negative existential proposition and S contains a definite description. Intuitively, $v(S) = T$ since the the object that the definite description refers to does not exist. However, in order for the definite description to be meaningful, it must refer to some unique thing in the domain. And, if it refers to something in the domain, then that thing exists. And so, $v(S) = F$, which means that non-existent entities really do exist.

A third problem with treating definite descriptions as referring (denoting) phrases involves the substitutivity principle. This principle says that co-referring expression (i.e., expressions that refer to the same thing) can be substituted for each other without changing the truth-value of the wff in which the substitution is made. For example, if Pa and $a = b$, then Pb. However, consider the following proposition:

1. Tek wanted to know if Scott was the author of Waverley.

Let's suppose that (1) is true. Now consider the following argument:

1. Tek wants to know if Scott was the author of Waverley.
2. Scott = the author of Waverley.
3. Therefore, Tek wants to know if Scott is Scott.

Thus, treating definite descriptions as though they were referring phrases seems to require us to reject the substitutivity principle.

The problem in the above example is that if the principle of substitutivity is true, then given the identity of Scott and the author of Waverley, we can substitute "Scott" for "the author of Waverley" in (1). This leads to the false conclusion that Tek wonders whether Scott is identical to Scott.

From here we have at least two options. First, we might take the above as a genuine analysis of definite descriptions and accept that the principle of bivalence is not universally true. Were we to accept this option, we might either (i) point out a certain limitation of logical analysis or (ii) propose an alternative (deviation) from classical logic. Along the lines of (ii), one option would be to pursue a multi-valued approach to logic, where some wffs given the truth value of "neither true nor false" or "indeterminate". Second, we might try to take the above as a faulty analysis of definite descriptions. Rather than giving up on the possibility of logically analyzing sentences with definite descriptions or giving up on the universality of the principle of bivalence, we might instead resort to a different analysis than what is given above. In what follows, we consider two possible approaches that aim to logically treat definite descriptions while aiming to preserve the principle of bivalence.

The first approach we consider is credited to Gottlob Frege. This approach begins by pointing out that there is a deficiency in natural language insofar as it allows for expressions that fail to refer to things. In particular, definite descriptions in natural languages allow for speakers to express content that aims to refer to a unique thing which either does not exist or is not unique. This deficiency can be overcome with a better, more ideal language. In such a language, any use of a definite description would always refer to something. Frege's solution then was to posit the existence of a special object called the nil entity, which definite descriptions would refer to when they failed to pick out a unique object. Doing this would allow for assigning sentences containing a definite description of a truth value of T or F.On this approach, we revise the interpretation of $[\iota x\phi]_{\mathcal{D},g}$ as follows:

On the condition there exists a unique individual $u \in \mathcal{D}$, $[\iota x\phi]_{\mathcal{D},g}$ is that unique individual $u \in \mathcal{D}$ such that $v_{\mathcal{M},g^x_u}\phi = T$, otherwise $[\iota x\phi]_{\mathcal{D},g} = u_0$ where u_0 is the nil individual.

To illustrate, suppose again we are evaluating the sentence "The King of the United States is bald". On the Fregean analysis, since there is no King of the United States, the definite description "the King of the United States" refers to the nil individual. And, assuming that the nil individual is not bald (not found in the interpretation of B), it follows that the sentence is false, viz., $v(B(\iota x K x)) = F$. So, in short, on Frege's analysis, sentences involving definite descriptions that fail to refer to a unique object are false. And, in calling these sentences false rather than neither true nor false, the principle of bivalence is preserved.

The second approach we consider is due to Bertrand Russell. Similar to Frege's approach, (i) the principle of bivalence is preserved, (ii) the underyling problem with definite descriptions is due to a deficiency in natural language, and (iii) the proper correction to these problems is through an ideal (logical) language.

The supposed problem with natural language is that definite descriptions are treated as though they were singular terms (names). Roughly speaking, definite descriptions and proper names both appear to have the same syntactic role in that they are the subjects of sentences. For instance, compare the following:

1. Donald Trump is the President of the United States
2. The author of the *Art of the Deal* is the President of the United States.

In (1) and (2), the proper name and the definite description both play the role of the subject and receive the same predicate. However, Russell argued that to treat definite descriptions as names would be to mistakenly read the grammatical form of the sentence into its logical form (a mistake similar to reading "everyone/someone" as though they referred to some object rather than to treat them as quantificational expressions).

The way forward is not, however, to directly define the iota operator but instead develop a method such that for any wff containing the iota operator (a definite description), an equivalent RL-wff without the iota operator can be provided.

Let us return then to the sentence "the King of the United States is bald". This sentence was translated as $B(\iota x K x)$. According to Russell, sentences of this type make three claims:

1. an existence claim: $(\exists x)Kx$, viz., there exists an x such that x is the King of the United States
2. a singularity (or uniqueness) claim: $(\forall y)(Ky \rightarrow y = x)$, viz., for any individual y, if that individual is King of the United States, then that individual is x
3. a predication claim: Bx, viz., x is bald.

Putting all three of the above claims together, "the King of the United States is bald" can be translated as $(\exists x)(Kx \wedge (\forall y)(Ky \rightarrow y = x) \wedge Bx)$. That is, there is exactly one individual that has the property of being King of the United States and that individual is bald.

Given this treatment of definite descriptions, "the present King of the United States is bald" can be false in at least three different ways. First, there can fail to be an individual that is the King of the United States. Second, there can fail to be a single (exactly one) individual that is the King of the United States. Third, the single individual that is the King of the United States may not be bald. And so, Russell has a way to treat definite descriptions while preserving the principle of bivalence. This is done by treating sentences with definite descriptions as complex quantificational expressions.

Russell's treatment of definite descriptions also illustrates an ambiguity found in negative sentences involving said descriptions. For instance, consider the sentence "the present King of the

United States is not bald". This sentence is ambiguous. It says one of the following:

1. there does not exist a present King of the United States that is bald
2. there exists a present King of the United States but this individual is not bald

Each of these sentences can be translated using Russell's approach:

1. $\neg(\exists x)(Kx \wedge (\forall y)(Ky \rightarrow y = x) \wedge Bx)$
2. $(\exists x)(Kx \wedge (\forall y)(Ky \rightarrow y = x) \wedge \neg Bx)$

Exercise 1-8: *Translate the following wffs into the language of predicate logic using the following translation key:* j : *John,* s : *Sally,* Ax : x *is angry,* Dx : x *is a dictator,* Lxy : x *loves* y

1. John is angry
2. The man is angry
3. The president is a dictator
4. The man is identical to the president.
5. John loves the dictator.
6. The dictator loves John.