Completeness of Propositional Logic via Maximal Consistent Sets (The Lindenbaum-Henkin Method)

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Theorem (Strong completeness for PL). For any set of wffs Γ and formula φ, if $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

This says *if there is no interpretation (model) where the members of* Γ *are true and* φ *is false, then there exists a formal proof of* φ *from* Γ. In propositional logic (PL), a model $\mathcal{M}$ is an interpretation $\mathscr{I}$ of the propositional letters. So, we can rephrase the above as saying *if there is no model where the members of* Γ *are true and* φ *is false, then there exists a formal proof of* φ *from* Γ.

Definitions and rules

First, some definitions.

- The language of propositional logic $\mathcal{L}$ is a set of letters $P, Q, R, \ldots$ and two truth-functional operators: $\neg, \rightarrow$.
- Well-formed formulas (wffs) are built from propositional letters $P, Q, R, \ldots$ using $\neg$ and $\rightarrow$.
- Let $\perp$ be "an outright contradiction," e.g. $\neg(P \rightarrow P)$. It is a *syntactic* notion: it just means some specific unprovable-looking wff that is the negation of a theorem.
- $\Gamma \vdash \varphi$ means "there is a finite formal derivation of φ from the premises in Γ," using some fixed set of axioms (e.g. the three Hilbert axioms) and modus ponens ($\rightarrow E$).
- $\mathscr{I}$ is an interpretation: an assignment of 1 (true) or 0 (false) to the every propositional letter $P, Q, R \ldots$ in the language. An $\mathscr{I}$ is also known as a truth-value assignment to the propositional letters.
- $\mathcal{M}$ is a model, a one-part structure consisting of an interpretation of the proposition letters.
- A set of wffs Γ is satisfiable if there is an interpretation (truth assignment in propositional logic) under which every wff in Γ is true.
- $\Gamma \models \varphi$ means "every interpretation $\mathscr{I}$ that makes every element of Γ true is also an interpretation $\mathscr{I}$ that makes φ true." Equivalently, $\Gamma \models \varphi$ iff there is no interpretation that makes every member of Γ true and φ false.

We'll assume the following Hilbert axiom schemas, where A, B, and C range over all wffs:

(A1) $A \to (B \to A)$,
(A2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,
(A3) $(\neg A \to \neg B) \to ((\neg A \to B) \to A)$.

Finally, we'll use the following rules:

- **Deduction theorem:** $\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \to \psi$.
- **Double negation:** $\vdash \neg\neg\varphi \to \varphi$ and $\vdash \varphi \to \neg\neg\varphi$.
- **Ex falso quodlibet:** $\varphi, \neg\varphi \vdash \psi$ or $\bot \vdash \psi$ (from a contradiction, anything follows).
- **Negated conditional:** $\neg(\varphi \to \psi) \vdash \varphi$ and $\neg(\varphi \to \psi) \vdash \neg\psi$.
- **Contraposition:** $\varphi \to \psi \dashv\vdash \neg\psi \to \neg\varphi$.
- **Monotonicity of derivability**: if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$. Intuitively, if we can derive φ from Γ, then we can derive φ from a potentially larger set Δ. In other words, adding premises to a proof of ϕ does not undermine the proof.

Consistency and Maximality

Let's define when a set of wffs Γ is (1) inconsistent, (2) consistent, (3) maximal, and (4) maximal consistent.

1. **Definition (inconsistent).** Γ is **inconsistent** iff $\Gamma \vdash \bot$.
2. **Definition (consistent).** Γ is **consistent** iff it is not inconsistent ($\Gamma \nvdash \bot$).
3. **Definition (maximal).** Γ is **maximal** iff for every wff φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
4. **Definition (maximal consistent).** Γ is **maximal consistent** iff (i) for every wff φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ and (ii) Γ is consistent.

It is helpful to have some intuitions surrounding these ideas.

First, you can think of an inconsistent set as a story where you can prove contains a contradiction. That is, it is a story that you can show that "the butler was the murderer and not the murderer." In contrast, a consistent set can be thought of as a story that you cannot prove a contradiction.

Second, for many consistent stories, you can add more details to the story without it becoming inconsistent *as long as you're careful about what you add.* You don't want to add that the main character is both tall and not-tall because then the story would contain a proposition P and its literal negation $\neg P$.

Third, a **maximal** set is a story that is fully complete: for every possible sentence in the language, the story either asserts it or asserts its negation. Introduce a character into the story: well, the character has brown hair or does not. You can ask a maximal set story any "yes" or "no" question and the story has an answer.

Fourth, a **maximal consistent** set is that is both maximal (complete, decides everything) and consistent (no contradictions).

Structure of proof

The flow of our proof for completeness is as follows:

1. **Prove Toolbox Lemmas**: prove several lemmas we'll use to prove more important lemmas.
2. **Prove Lindenbaum's Lemma:** show that from a consistent set Δ, there exists a maximal consistent set Γ that extends Δ.
3. **Prove properties of Γ:** Show Γ's membership rules mirror the operators – e.g. $(\varphi \to \psi) \in \Gamma$ iff $\varphi \notin \Gamma$ or $\psi \in \Gamma$.
4. Canonical valuation. Build an actual interpretation out of the maximal set
5. **Prove the Truth Lemma:** use those membership rules to prove, by induction on formula structure, that $v(\varphi) = T \iff \varphi \in \Gamma$ for **every** formula φ, not just atoms.
6. **Prove Strong completeness**. Package everything into the completeness theorem.

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3. Toolbox Lemmas

In this section, we'll prove several lemmas.

Lemma A (Finiteness / Compactness of $\vdash$)

Lemma A. $\Gamma \vdash \varphi \iff$ for some finite $\Gamma' \subseteq \Gamma, \Gamma' \vdash \varphi$.

Proof ($\Rightarrow$). Assume $\Gamma \vdash \varphi$. A derivation is, by definition, a *finite* sequence of formulas. So, regardless of whether Γ is finite or infinite, any derivation of φ from Γ will only consist of a finite number of wffs from Γ. Let Γ' be that set. And so, $\Gamma' \vdash \varphi$.

More intuitively, a proof is a finite document. As a finite document, it can only make use of a finite number of sentences from an infinite number of possible sentences.

Proof ($\Leftarrow$). Assume that for some finite $\Gamma' \subseteq \Gamma, \Gamma' \vdash \varphi$. By monotonicity $\Gamma \vdash \varphi$.

Lemma B (Explosion from contradiction)

Lemma B. If $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, then Γ is inconsistent.

Proof. Assume $\Gamma \vdash \varphi, \neg\varphi$. By ex falso quodlibet, $\varphi, \neg\varphi \vdash \bot$. By transitivity, $\Gamma \vdash \bot$.

Lemma DN (Double Negation)

> **Lemma DN**. If $\Gamma \cup \{\neg\varphi\} \vdash \bot$, then $\Gamma \vdash \varphi$.

Proof. Let $\bot$ abbreviate to $\neg(P \to P)$.

Lemma C (The pivot lemma)

> **Lemma C**. $\Gamma \vdash \varphi \iff \Gamma \cup \{\neg\varphi\}$ is inconsistent.

Proof ($\Rightarrow$): First, if $\Gamma \vdash \varphi$, then $\Gamma \cup \{\neg\varphi\} \vdash \varphi$ by monotonicity (adding a premise to Γ never undermines the proof). Second, trivially $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$. So:

1. $\Gamma \cup \{\neg\varphi\} \vdash \varphi$,
2. $\Gamma \cup \{\neg\varphi\} \vdash \neg\varphi$
3. By (1), (2), and Lemma B it follows that $\Gamma \cup \{\neg\varphi\}$ is inconsistent.

Proof ($\Leftarrow$): Since $\Gamma \cup \{\neg\varphi\}$ is inconsistent, it proves *everything* (ex falso). So, for an arbitrary wff ψ, $\Gamma \cup \{\neg\varphi\} \vdash \neg\psi$ and $\Gamma \cup \{\neg\varphi\} \vdash \psi$. By the deductive theorem, $\Gamma \vdash \neg\varphi \to \neg\psi$ and $\Gamma \vdash \neg\varphi \to \psi$.

Next, let's instantiate (A3) as follows: $A = \varphi$, and $B = \psi$:

1. $(\neg A \to \neg B) \to ((\neg A \to B) \to A)$
2. $(\neg\varphi \to \neg B) \to ((\neg\varphi \to B) \to \varphi)$
3. $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$

Next, we can reason as follows:

1. $(\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$
2. $\Gamma \vdash \neg\varphi \to \neg\psi$
3. $\Gamma \vdash \neg\varphi \to \psi$
4. $\Gamma \vdash (\neg\varphi \to \psi) \to \varphi$. (1, 2, modus ponens)
5. $\Gamma \vdash \varphi$ (3, 4 modus ponens)

Therefore, $\Gamma \vdash \varphi$. ■

Lemma C allows us to swap between there being a **proof** of φ from Γ and there being an **inconsistency** between Γ and $\neg\varphi$ (the negation of the conclusion).

Lemma D (Consistent unions of chains)

> **Lemma D**. If $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is an increasing chain of consistent sets, then $\Gamma = \bigcup_n \Gamma_n$ is consistent.

Proof. Suppose $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is an increasing chain of consistent sets, but $\Gamma = \bigcup_n \Gamma_n$ is **inconsistent**: $\Gamma \vdash \bot$.[1]

By Lemma A, there is some *finite* $\Gamma' = \{\varphi_1, \ldots, \varphi_k\} \subseteq \Gamma$ with $\Gamma' \vdash \bot$. Because Γ' is finite, each formula in Γ' belongs to some stage Γ_{n_i} in the increasing chain.

[1] By "chain", what is meant is a collection of sets ordered by inclusion: $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$. Each stage contains everything from the prior stage, possibly with some additional wffs.

Let N be the largest of these finitely many indices: $N = max(0, n_1, \ldots, n_k)$. Since the chain is nested, $\Gamma_{n_i} \subseteq \Gamma_N$ for each i, $\Gamma' \subseteq \Gamma_N$. By monotonicity of derivability, since $\Gamma' \vdash \bot$, it also follows that $\Gamma_N \vdash \bot$. And since $\Gamma_N \vdash \bot$, it follows that Γ_N is inconsistent.

It thus follows that if the union of the sets is inconsistent, then some member of the chain is inconsistent. Equivalently, if every $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ is consistent, then the union is consistent.■

Intuition: If a set is inconsistent, there needs to be a concrete, finite contradiction. If no finite stage of the chain ever contains a contradiction, the infinite union can't suddenly contain one either.

Lemma E (The branching fact)

> **Lemma E**: If Γ is consistent, then for any formula φ, $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{\neg\varphi\}$ is consistent (or both).

Proof: Suppose, for contradiction, that Γ is consistent and *both* $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent.

1. If $\Gamma \cup \{\varphi\}$ is inconsistent $\Rightarrow \Gamma \vdash \neg\varphi$. (Lemma C)
2. If $\Gamma \cup \{\neg\varphi\}$ is inconsistent $\Rightarrow \Gamma \vdash \varphi$. (Lemma C)
3. Given $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, Γ is inconsistent. (Lemma B)

Since Γ is inconsistent contradicts our assumption that Γ is consistent, Lemma E is the case. ■

Lindenbaum's Lemma (Maximal Consistent Extension)

Next, let's prove Lindenbaum's Lemma.

> **Lemma (Lindenbaum).** Every consistent set of formulas Δ can be extended to a maximal consistent set Γ (where $\Delta \subseteq \Gamma$).

Intuitive Idea behind the proof

Think of Δ as a book of consistent but incomplete claims about some imagined world. Since the book is consistent, there are no contradictions. It does not say "Tek is a jerk" and "Tek is not a jerk". Since the notebook is incomplete, the book does not tell us everything about the world. For example, it does not tell us the color of Tek's hair.

Now, suppose you have an infinite amount of time so you want to turn the incomplete book to a complete book by saying everything there is to say about the world. You could do this by just writing every proposition. If you did this, Δ would go from consistent and incomplete to inconsistent and complete. But

this is not what you want. You want to transform Δ (our book) from incomplete to complete but you also want Δ to remain consistent.

How can you achieve this goal? Your process of creating this notebook works like this:

First, line up every sentence in Δ a list $S_1, S_2, S_3, \ldots$. Second, go through each wff, one at a time, and for each S_n, ask:

> "Can I add S_n to my notebook without my notebook (Δ) becoming inconsistent?"

- If **yes**, add S_n.
- If **no**, add $\neg S_n$ instead.

By Lemma E, recall that we will always be able to add at least one of these sentences without turning our notebook from consistent to inconsistent.

Formal proof

Let Δ be a consistent set of wffs. The goal is to show that Δ can be extended to a maximal, consistent set Γ.

Let's suppose we can enumerate **every** wff of $\mathcal{L}$ as $\varphi_1, \varphi_2, \varphi_3, \ldots$. Now, let's define a chain of sets:

$$\Gamma_0 = \Delta, \qquad \Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_{n+1}\} & \text{if this is consistent} \\ \Gamma_n \cup \{\neg\varphi_{n+1}\} & \text{otherwise} \end{cases}$$

Finally, let

$$\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n.$$

Next, let's prove that for any number n, Γ_n is consistent. We will prove this via induction. After that, we will prove Γ is consistent.

Base case. $\Gamma_0 = \Delta$, which is consistent by hypothesis.

Inductive step. Assume that for some n, Γ_n is consistent. We must show Γ_{n+1} is consistent.

- Lemma E guarantees that *at least one* of $\Gamma_n \cup \{\varphi_{n+1}\}$ or $\Gamma_n \cup \{\neg\varphi_{n+1}\}$ is consistent.
- If $\Gamma_n \cup \{\varphi_{n+1}\}$ is *consistent*, then $\Gamma_{n+1} = \Gamma_n \cup \{\varphi_{n+1}\}$, and so Γ_{n+1} is consistent.
- If $\Gamma_n \cup \{\varphi_{n+1}\}$ is *not consistent*, then since Γ_n is consistent (IH), Lemma E guarantees $\Gamma_n \cup \{\neg\varphi_{n+1}\}$ is consistent. Set Γ_{n+1} to $\Gamma_n \cup \{\neg\varphi_{n+1}\}$.

So in both branches, Γ_{n+1} is consistent. By induction, every Γ_n is consistent. ■

Next, let's prove that Γ is consistent.

Proof. $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \Gamma_n$ is an increasing chain of consistent sets (just proved). Let Γ be its union. By Lemma D, Γ is consistent.

Now that we have proven that Γ is consistent, let's prove that Γ is maximal.

Proof. Let φ_i be *any* wff of the language. Since our enumeration $\varphi_1, \varphi_2, \ldots$ lists *every* wff, $\varphi_i = \varphi_n$ for some n. So, via the construction Γ_n, either $\varphi_i \in \Gamma_n$ or $\neg\varphi_i \in \Gamma_n$: one of the two was inserted at stage n. Since $\Gamma_n \subseteq \Gamma$, we get $\varphi_i \in \Gamma$ or $\neg\varphi_i \in \Gamma$. And since φ_i was an arbitrary wff, Γ is maximal. ■

Recall that Lindenbaum's Lemma states that every consistent set of formulas Δ can be extended to a maximal consistent set Γ (i.e., $\Delta \subseteq \Gamma$). What has been shown is (1) Γ contains Δ (since $\Delta = \Gamma_0 \subseteq \Gamma$), Γ is consistent, and Γ is maximal. This proves Lindenbaum's Lemma.

A worked mini-example

Suppose our language $\mathcal{L}$ only has two letters, P, Q. Let $\Delta = \{P\}$. It is obvious that Δ is consistent. Enumerate the first few formulas as:

$$\varphi_1 = Q, \quad \varphi_2 = (P \to Q), \quad \varphi_3 = \neg Q, \quad \varphi_4 = (Q \to P), \ldots$$

Next, let's build a consistent, maximal set from Δ.

- $\Gamma_0 = \{P\}$.
- **Stage 1:** Is $\{P, Q\}$ consistent? Yes. $\Gamma_1 = \{P, Q\}$.
- **Stage 2:** Is $\{P, Q, (P \to Q)\}$ consistent? Yes. $\Gamma_2 = \{P, Q, P \to Q\}$.
- **Stage 3:** Is $\{P, Q, P \to Q, \neg Q\}$ consistent? No. Since we have Q, adding $\neg Q$ to Γ_2 would allow for $\Gamma_2 \vdash Q, \neg Q$, which, by Lemma B, would result in Γ_2 being inconsistent. So instead of adding of adding φ_3 ($\neg Q$), add $\neg\varphi_3 = \neg\neg Q$: $\Gamma_3 = \{P, Q, P \to Q, \neg\neg Q\}$. The result is consistent via Lemma E.
- **Stage 4:** Is adding $(Q \to P)$ consistent? Yes, it can be proven from Γ_3, and so $\Gamma_4 = \Gamma_3 \cup \{Q \to P\}$.
- and so on through every wff of $\mathcal{L}$.

Properties of Maximal Consistent Sets

Having built a maximal consistent set Γ, we now show it behaves exactly like a valuation of propositional wffs.

> **Corollary 1:** For a maximal consistent Γ, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, **but not both**.

Proof. Maximality gives "at least one." If both were in Γ, then $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$ (premises are always derivable from themselves), so by Lemma B, Γ is inconsistent. ■

Note: Note that Γ is similar to truth-value assignment in that (1) either a wff is in Γ or it is not, while (2) a wff is either true or false, not both.

Corollary 2 (Deductive closure): If $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.

In other words, a maximal consistent set already contains everything it can prove. That is, if φ is provable from Γ, then it is an element of Γ.

Proof. For reductio, assume $\Gamma \vdash \varphi$ and $\varphi \notin \Gamma$. By Corollary 1, since Γ is maximal, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. Since $\varphi \notin \Gamma$, $\neg\varphi \in \Gamma$. Since $\neg\varphi \in \Gamma$, $\Gamma \cup \neg\varphi = \Gamma$ (adding an existing member changes nothing).

Γ is consistent because it is a maximal consistent set. By Lemma C, $\Gamma \vdash \varphi \Rightarrow \Gamma \cup \{\neg\varphi\}$ is inconsistent. Its contrapositive is $\Gamma \cup \{\neg\varphi\}$ is consistent $\Rightarrow \Gamma \nvdash \varphi$. Since $\Gamma \cup \{\neg\varphi\}$ is consistent, it follows that $\Gamma \nvdash \varphi$, which contradicts the assumption $\Gamma \vdash \varphi$. Therefore, $\varphi \in \Gamma$. ■

Intuition: Γ is "saturated": anything it logically entails, it already explicitly contains.

Corollary 3 (The conditional clause): For maximal consistent Γ:
$(\varphi \to \psi) \in \Gamma \iff \varphi \notin \Gamma$ or $\psi \in \Gamma$.

Proof ($\Rightarrow$). Suppose $\varphi \to \psi \in \Gamma$ but (for contradiction) $\varphi \in \Gamma$ and $\psi \notin \Gamma$. From $\varphi \to \psi$ and φ, modus ponens gives $\Gamma \vdash \psi$, so by Corollary 2, $\psi \in \Gamma$ — contradicting $\psi \notin \Gamma$.

Proof ($\Leftarrow$). Suppose $\varphi \notin \Gamma$ or $\psi \in \Gamma$, but (for contradiction) $\varphi \to \psi \notin \Gamma$. By Corollary 1, $\neg(\varphi \to \psi) \in \Gamma$. The "negated conditional" fact gives $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\psi$. By Corollary 2, $\varphi \in \Gamma$ and $\neg\psi \in \Gamma$ (so $\psi \notin \Gamma$, by Corollary 1). This contradicts our assumption ("$\varphi \notin \Gamma$ or $\psi \in \Gamma$") either way. ■

Intuition: this is exactly the truth table for $\to$, rewritten in terms of set membership instead of truth values: "$\varphi \to \psi$ is true" is equivalent to "φ is false or ψ is true." Corollary 3 shows the *syntactic* object Γ mirrors this *semantic* pattern exactly — which is precisely what lets us turn Γ into an actual valuation in the next section.

The Canonical Valuation and the Truth Lemma

Building a valuation out of Γ

Define an interpretation $\mathscr{I}$ directly from the maximal consistent set Γ:

$$\mathscr{I}(P) = T \iff P \in \Gamma \quad \text{(for every propositional letter } P\text{)}.$$

Next, a valuation (called a "canonical valuation") can be derived from this interpretation:

$$v(P) = T \iff \mathscr{I}(P) = T(\text{for every atomic wff } P).$$

With this established, **membership in Γ becomes truth under v.**

$$v(P) = T \iff P \in \Gamma \quad (\text{for every propositional letter } P).$$

6.2 Truth Lemma

Next, we wish to show not merely that $v(P) = T \iff P \in \Gamma$ for every letter P, but $v(\varphi) = T \iff \varphi \in \Gamma$ for every wff φ. Let's call this the "Truth Lemma":

> **Truth Lemma.** For *every* formula φ (not just atomic letters): $v(\varphi) = T \iff \varphi \in \Gamma$.

Proof, by induction on the structure (number of operators) of φ.

First, there is the base case. In the base case, φ is an atomic wff. Since the atomic wff is just a letter, $v(\varphi) = T \iff \varphi \in \Gamma$ because $v(P) = T \iff P \in \Gamma$.

Second, there are negated wffs ($\varphi = \neg\gamma$). Assume the lemma holds for γ (induction hypothesis = IH): $v(\gamma) = T \iff \gamma \in \Gamma$.

- $v(\neg\gamma) = T \iff v(\gamma) = F$ via the truth table for negation.
- By IH, $v(\gamma) = T \iff \gamma \in \Gamma$. And so, $v(\gamma) = F \iff \gamma \notin \Gamma$.
- By Corollary 1, since Γ is a maximal consistent set, $\gamma \notin \Gamma \iff \neg\gamma \in \Gamma$
- And so, $v(\neg\gamma) = T \iff \neg\gamma \in \Gamma$

Third, there is the inductive step for conditionals ($\varphi = \gamma \to \psi$). Again, assume the lemma holds for γ and for ψ (IH): $v(\gamma) = T \iff \gamma \in \Gamma$ and $v(\psi) = T \iff \psi \in \Gamma$.

- $v(\gamma \to \psi) = T \iff v(\gamma) = F$ or $v(\psi) = T$ via the truth table for conditionals.
- $v(\gamma) = F$ or $v(\psi) = T \iff \gamma \notin \Gamma$ or $\psi \in \Gamma$ by IH
- $\gamma \notin \Gamma$ or $\psi \in \Gamma \iff (\gamma \to \psi) \in \Gamma$ by Corollary 3.

Since every operator (negation, conditional) is covered, the Truth Lemma holds for all formulas. ■

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7. Model Existence Lemma

> **Lemma.** Every consistent set has a model, where a model of Δ is an interpretation $\mathscr{I}$ with $v_{\mathscr{I}}(\varphi) = T$ for all $\varphi \in \Delta$.

A model $\mathcal{M}$ of a set of wffs Σ is an interpretation $\mathscr{I}$ (or the valuation v it induces) that makes every member of Σ true.

Proof. Let Δ be a consistent set. By **Lindenbaum's Lemma**, Δ is extendable to a maximal consistent set $\Gamma \supseteq \Delta$. Build the canonical valuation v from Γ as above. By the **Truth Lemma**, $v(\varphi) = T$ for every $\varphi \in \Gamma$. Since v makes every member of Γ true, and $\Gamma \supseteq \Delta$, v make every member of Δ true as well. And so, v is a model of Δ. ■

This is the crux fact: **consistency (a syntactic, proof-theoretic property) guarantees satisfiability (a semantic, model-theoretic property).** Soundness already gave us the reverse direction (satisfiable ⇒ consistent, roughly); Model Existence is what completeness needs.

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8. The Completeness Theorem

Let's bring together our earlier results to prove strong completeness:

> **Theorem (Strong completeness).** If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Proof.

1. Assume $\Gamma \models \varphi$.
2. Then no valuation makes every member of Γ true while making φ false. Equivalently, **no valuation satisfies** $\Gamma \cup \{\neg\varphi\}$.
3. By the contrapositive of **Model Existence**: if a set has no model, it is not consistent, i.e. it is inconsistent. So $\Gamma \cup \{\neg\varphi\}$ is inconsistent.
4. By Lemma C, $\Gamma \cup \{\neg\varphi\}$ is inconsistent $\iff \Gamma \vdash \varphi$.
5. Therefore $\Gamma \vdash \varphi$. ■

Strong completeness has been proven. As a corollorry, weak completeness is also proven.

Corollary (Weak completeness): If $\models \varphi$, then $\vdash \varphi$.

This says that every tautology (wff true under every interpretation) is a theorem (provable from axioms / assumptions alone).

Proof. This is a special case of strong completeness, when $\Gamma = \varnothing$: if $\models \varphi$ (i.e. φ is a tautology), then $\vdash \varphi$ (i.e. φ is a theorem). ■

Summary

We have proven **strong completeness** ($\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$) and **weak completeness** ($\models \varphi \Rightarrow \vdash \varphi$). The former asserts that if a set of wffs semantically entails a formula, then there is a proof from that set to that formula. More casually,

it asserts that every valid argument has a proof. The latter asserts that every tautology is a theorem.