Handout 8

RL Derivations

In this chapter, we specify a deductive apparatus for **RL**.

> DEFINITION – DEDUCTIVE APPARATUS
>
> A deductive apparatus for **RL** is a set of rules of derivation (or "inference" rules) that express which wffs **Q** can be written after which wffs **P** in a derivation. The deductive apparatus for **RL** is hereafter abbreviated as **RD**.

> DEFINITION – DERIVATION OF **Q** IN **RD**
>
> A derivation in **PD** of **Q** is a finite (not infinite and not empty) string of formulas from a set Γ of **RL** wffs where (i) the last formula in the string is Q and (ii) each formula is either a premise, an assumption, or is the result of the preceding formulas and the deductive apparatus.

In this handout, we add to the existing intelim rules formulated for **PD** with a set of intelim rules for quantified expressions. In addition, we present derivations vertically, number each wff in the derivation, and make use of vertical lines to indicate the scope of any assumptions that are made. In addition, each wff in the derivation is justified by citing both a rule from **PD** and the line numbers of the wffs that are necessary for the application of said rule.

If there is a derivation for a wff **Q** from a set of wffs Γ, then we say that **Q** is a syntactic consequence (or syntactically entailed by) Γ.

> DEFINITION – SYNTACTIC CONSEQUENCE
>
> A formula **Q** is a syntactic consequence in **RD** of a set Γ of **RL** wffs if and only if there is a derivation in **RD** of Q from Γ. To express that **Q** is a syntactic consequence of Γ, we write $\Gamma \vdash Q$.

8.1 Predicate Derivations: Four Quantifier Rules

The proof system for **RL** is RD, which consists of $PD+$ (the proof system from propositional logic) and four new additional derivation rules for quantified formulas.

Derivation Rule – Universal Elimination ($\forall E$)

From any universally quantified proposition $(\forall x)P$, we can derive a substitution instance $P(a/x)$ in which all bound variables are consistently replaced with names.

$(\forall x)P \vdash P(a/x)$

The idea is that if you have a universally quantified proposition $(\forall x)P$, you can move forward a step in the proof with a proposition $P(a/x)$ that is the result of removing the universal quantifier and replacing any bound variables with any name of your choosing.

Example 3:

1	$(\forall x)Px$	P
2	Pa	$\forall E$, 1
3	Pb	$\forall E$, 1
4	Pe	$\forall E$, 1

When using $\forall E$, replacement of variables with names must be uniform.

Example 4:

1	$(\forall x)Rxx$	P
2	Raa	$\forall E$, 1
3	Rbb	$\forall E$, 1
4	Rab	$\forall E$, NO!, 1

Derivation Rule – Existential Introduction ($\exists I$)

From any possible substitution instance $P(a/x)$, an existentially quantified proposition $(\exists x)P$ can be derived by consistently replacing at least one individual constant (name) with an existentially quantified variable.

$P(a/x) \vdash (\exists x)P$

The idea is that if you have a non-quantified proposition Pa, you can move forward a step in the proof with an existentially quantified proposition $(\exists x)P(x/a)$ that is the result of uniformly replacing at least one name with an existentially quantified bound variable.

Example 5:

1	Pa	P
2	Rbb	P
3	$(\exists x)Px$	$\exists I$, 1
4	$(\exists x)Rxx$	$\exists I$, 2

When using $\exists I$, replacement of variables with names must be uniform but you need only replace one name.

Example 6:

1	Lab	P
2	Rbb	P
3	$(\exists x)Lax$	$\exists I$, 1
4	$(\exists x)Lxb$	$\exists I$, 1
5	$(\exists x)Lxx$	1$\exists I$, NO!, 1
6	$(\exists x)Rxx$	$\exists I$, OK!, 2
7	$(\exists x)Rbx$	2$\exists I$, OK!, 2

Exercise 1: *Focusing on $\forall E$ and $\exists I$, solve the following proofs.*

1. $Paa \rightarrow Rbb, Pbb \rightarrow Rcc, (\forall x)Pxx \vdash Rbb \wedge Rcc$
2. $(\exists x)Rx \rightarrow (\exists x)Mx, Ra \vdash (\exists x)Mx$
3. $Pab, (\exists x)(\exists y)Pxy \rightarrow (\forall x)Zxx \vdash Zaa$
4. $(\forall x)(\forall y)Pxy,\ Paa \rightarrow Lab \vdash (\exists y)(\exists x)Lxy$

Derivation Rule – Universal Introduction ($\forall I$)

A universally quantified proposition $(\forall x)P$ can be derived from $P(a/x)$ provided (1) a does not occur as a premise or as an assumption in an open subproof, and (2) a does not occur in $(\forall x)P$.

$P(a/x) \vdash (\forall x)P$ (when a does not occur as premise, open subproof, or in $(\forall x)P$.

The idea is that if you have a non-quantified proposition Pa, you can move forward a step in the proof with a universally quantified proposition $(\forall x)P(x/a)$

that is the result of uniformly replacing each name with universally quantified bound variables. However, there are two restrictions you must be mindful of: (1) the names cannot occur in the premises or as the assumption in an open subproof and (2) the names cannot occur in the universally quantified proposition $(\forall x)P(x/a)$ derived.

Example 7:

1	$(\forall x)Pxx$	P
2	Paa	$\forall E$, 1
3	$(\forall y)Pyy$	$\forall I$, 2

Example 8:

1	$\quad \mid Raa$	A
2	$\quad \mid Raa$	R, 1
3	$Raa \rightarrow Raa$	$\rightarrow I$, 1–2
4	$(\forall x)(Rxx \rightarrow Rxx)$	$\forall I$, 3

In the above example, $(\forall I)$ is applied to line 3 even though 'a' is in the assumption. However, the subproof involving 'a' is no longer open.

Example 9: Violation of Restriction #1

1	Laa	P
2	$(\forall x)Lxx$	∀I, 1

Example 10: Second Violation of Restriction #1

1	$\mid Laa$	A
2	$\mid (\forall x)Lxx$	$\forall I$, NO!, 1

Example 11: Violation of Restriction #2

1	$(\forall x)Pxx$	P
2	Paa	∀I, 1
3	$(\forall y)Pya$	$\forall I$, NO!, 2

Derivation Rule – Existential Elimination ($\exists E$)

From an existentially quantified expression $(\exists x)P$, an expression Q can be derived from the derivation of an assumed substitution instance $P(a/x)$ of $(\exists x)P$ provided (1) the individuating constant a does not occur in any premise or in an active proof (or subproof) prior to its arbitrary introduction in the assumption $P(a/x)$ and (2) the individuating constant a does not occur in proposition Q discharged from the subproof.

1	$(\exists x)\mathbf{P}$	P
n	$\quad\vert\ \mathbf{P}(a/x)$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ \mathbf{Q}$	
(k)	$\mathbf{Q}$	$\exists E$, 1, n–$(n+1)$

The idea is that if you have an existentially quantified proposition $(\exists x)\mathbf{P}$, you can derive $\mathbf{Q}$ provided you assume $\mathbf{P}(a/x)$ then derive $\mathbf{Q}$ with that subproof. However, this derivation is only permitted when (1) 'a' from $P(a/x)$ does not occur in the premises or an active proof and (2) 'a' from $P(a/x)$ does not occur in Q.

Example 12:

1	$(\exists x)Px$	P
2	$\quad\vert\ Pa$	A
3	$\quad\vert\ (\exists y)Py$	$\exists I$, 2
4	$(\exists y)Py$	$\exists E$, 1, 2–3

Example 13: Violation of Restriction #1

1	$(\exists z)Wzz$	P
2	$Wbb \rightarrow Lc$	P
3	$\quad\vert\ Wbb$	A
4	$\quad\vert\ Lc$	$\rightarrow E$, 2, 3
5	Lc	$\exists E$, NO!, 1, 3–4

Example 14: Violation of Restriction #1

1	$(\forall z)(Pzz \rightarrow Lzz)$	P
2	$(\exists z)Pzz$	P
3	$\quad Pbb \rightarrow Wcc$	A
4	$\quad\quad Pbb$	A
5	$\quad\quad Wcc$	$\rightarrow E$, 3, 5

5Wcc

Example 15: Violation of Restriction #2

1	$(\exists z)Wzz$	P
2	$\quad Wbb$	A
3	$\quad (\exists x)Wbx$	$\exists I$, 2
4	$(\exists x)Wbx$	$\exists E$, NO!, 1, 2–3

Exercise 2: *Solve the following proofs. In doing so, keep in mind that you may have to use any of the four derivation rules from RD.*

1. $(\forall x)(\forall y)Lxy, (\forall x)Lxx \rightarrow (\exists x)(\exists y)Lxy \vdash (\exists x)(\exists y)Lxy$
2. $(\exists x)Pxx \vdash (\exists x)(Pxx \vee Rx)$
3. $(\exists x)(\exists y)Pxy \vdash (\exists y)(\exists z)Pyz$
4. $(\forall x)(Pxx \rightarrow Pxx) \rightarrow Sa \vdash Sa$
5. $(\forall x)(Pxx \rightarrow Pxx) \rightarrow (\exists x)Mx \vdash (\exists x)(Mx \wedge Mx)$
6. $(\exists x)(\exists y)(\exists z)(Lxy \wedge Lyz), (\exists x)(\exists y)Lxy \rightarrow (\forall z)(\forall y)Pzy \vdash (\forall x)Pxx$

8.1.1 Quantifier Negation (QN)

In addition to the four introduction and elimination rules for quantified propositions, we can add to RD an equivalence rule (or rule of replacement). This rule, known as *Quantifier Negation*, allows us to replace negated quantified subformulas with non-negated quantified subformulas, and vice versa.

Derivation Rule – Quantifier Negation (QN)

From a negated universally quantified expression $\neg(\forall x)P$, an existentially quantified expression $(\exists x)\neg P$ can be derived, and vice versa. Also, from a negated existentially quantified expression $\neg(\exists x)P$, a universally quantified expression $(\forall x)\neg P$ can be inferred, and vice versa.

$\neg(\forall x)P \dashv\vdash (\exists x)\neg P$
$\neg(\exists x)P \dashv\vdash (\forall x)\neg P$

Example 1: of QN

1	$\neg(\forall x)Px$	P
2	$(\exists x)\neg Px$	QN, 1
3	$\neg(\forall x)Px$	QN, 2

Since (QN) is a rule of replacement, it can be applied to subformula of wffs.

Example 2: of QN

1	$\neg(\forall x)Px \to Pa$	P
2	$(\exists x)\neg Px \to Pa$	QN, 1

Notice that QN is applied to the antecedent of line 1.

Exercise 3: *Using QN and the other four derivation rules, solve the following proofs.*

1. $(\forall x)Pxx \to (\exists x)Rx, (\forall x)\neg Rx \vdash (\exists x)\neg Pxx$
2. $(\forall x)\neg(\exists y)Pxy \to (\forall x)Zx, (\forall x)(\forall y)\neg Pxy \vdash (\forall x)Zx$
3. $\neg(\exists x)(\exists y)Pxy, (\forall x)(\forall y)\neg Pxy \to (\forall z)Zz \vdash (\forall z)Zz$
4. $\neg(\forall x)Px \vdash (\exists x)(\neg Px \vee Zx)$
5. $\neg(\forall x)(\forall z)Pxz, (\exists x)(\exists z)\neg Pxz \to (\forall z)Lz \vdash (\forall w)(Lw \vee Mw)$
6. $\vdash (\exists x)Px \to (\forall x)(Px \to Px)$