Handout 2

PL: Symbols, Syntax, Semantics, Translation

In this handout, a formal language called "the language of propositional logic" (**PL**) is formulated. The language of propositional logic consists of a set of symbols (its vocabulary) and a syntax (its grammar expressed in terms of a set of formation rules). In addition, a semantics for **PL** is given. The semantics for **PL** is not, strictly speaking, part of the formal language. It is, instead, an interpretation of the formal language, i.e., a specification of what the different symbols and formulas of the language mean.

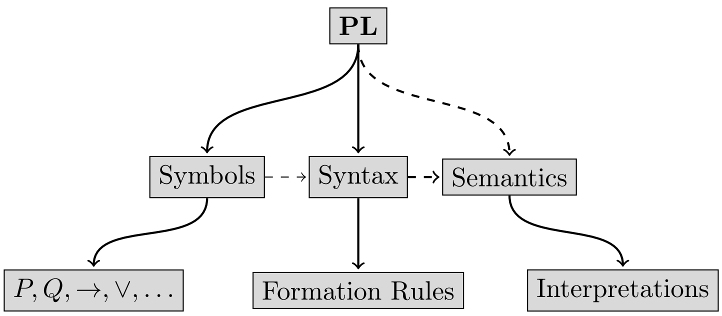

Figure 2.1 – A formal language consists of a set of symbols and a syntax. The semantics of a formal language concerns the interpretation of the symbols and/or well-formed formulas of that language.

2.1 PL: Symbols

PL consists of the following symbols:

1. an infinite number of "propositional letters": upper-case Roman (unbolded) letters with or without subscripted integers, e.g. $A_1, A_2, A_3, B, C, \ldots, Z$.

2. five truth-functional operators: $\vee, \rightarrow, \leftrightarrow, \neg, \wedge$
3. a left and right parenthesis: '(' and ')' to indicate the scope of truth-functional operators

1. Positive integers are subscripted to propositional letters to ensure that there are an infinite number of propositional letters available, e.g., $A_1, A_2, \ldots, A_n$
   (a) The use of 26 different Roman letters is not necessary. We could have easily formulated **PL** using a single letter P (and subscripted integers).
   (b) In fact, we could even formulate **PL** using binary notation (1s and 0s).
2. The five truth functional operators have a variety of different names:
   (a) $\neg$, "not" or "negation"
   (b) $\wedge$, "wedge" or "and"
   (c) $\vee$, "vee" or "or"
   (d) $\rightarrow$, "rightarrow"
   (e) $\leftrightarrow$, "doublearrow" or "if and only if" or "iff"

Exercise 1: *For each symbol below, indicate its type, viz., propositional letter, operator, etc.*

1. P
2. Q
3. $\wedge$
4. $\neg$
5. (
6. $\leftrightarrow$

2.2 PL: Syntax

The syntax of **PL** can be thought of as its grammar. It specifies which symbols can be put together in what order. The syntax of **PL** consists of a set of rules known as "formation rules". These formation rules specify which combinations of symbols is and is not a well-formed formula (abbreviated as 'wff', pronounced 'woof') in **PL**.

DEFINITION – WELL-FORMED FORMULA IN **PL**

A well-formed formula (or wff) is any formula that is capable of being generated by some combination of the seven formation rules in Table 2.1.

1	Every propositional letter of **PL** (e.g. A, B, C) is a wff.
2	If $\mathbf{P}$ is a wff, then $\neg(\mathbf{P})$ is a wff.
3	If $\mathbf{P}$ and $\mathbf{R}$ are wffs, then $(\mathbf{P} \wedge \mathbf{R})$ is a wff.
4	If $\mathbf{P}$ and $\mathbf{R}$ are wffs, then $(\mathbf{P} \vee \mathbf{R})$ is a wff.
5	If $\mathbf{P}$ and $\mathbf{R}$ are wffs, then $(\mathbf{P} \rightarrow \mathbf{R})$ is a wff.
6	If $\mathbf{P}$ and $\mathbf{R}$ are wffs, then $(\mathbf{P} \leftrightarrow \mathbf{R})$ is a wff.
7	Nothing else is a wff except what can be formed by repeated applications of 1–6.

Table 2.1 – Formation Rules for **PL**, where **P** and **R** are metavariables in **PL**

Example 1: The following are wffs in PL

1. P
2. $(P \rightarrow Q)$
3. $\neg((P \vee \neg(Q)))$
4. $\neg(\neg(P \wedge \neg(Q)))$

Example 2: The following are not wffs in PL

1. $P\neg$
2. PQ
3. $\vee\neg(Q))$
4. $\neg\neg P \wedge \neg Q \vee S$

To illustrate how the formation rules in Table 2.1 are used to determine whether a group of **PL** symbols are a wff, consider the following use of formation rules to show that $\neg(P) \rightarrow R$ is a wff:

1	Every propositional letter is a wff, so P and R are wffs.	(Rule 1)
2	If P is a wff, then $\neg(P)$ is a wff.	(Line 1 + Rule 2)
3	If $\neg(P)$ and R are wffs, then $(\neg(P) \rightarrow R)$ is a wff.	(Lines 1,2 + Rule 5)

Exercise 2: *Use the formation rules to show that the following formulas are wffs.*

1. $\neg(P)$
2. $(P \wedge \neg(R))$
3. $\neg(P \rightarrow \neg R)$

2.2.1 Kinds of Wffs

Here we specify three types of **PL** wffs.

DEFINITION – ATOMIC WFF

An atomic wff in **PL** is a well-formed formula in **PL** consisting only of a single propositional letter, e.g. P, R, S, D.

DEFINITION – COMPLEX WFF

A complex wff in **PL** is a well-formed formula in **PL** that contains at least one propositional letter and a truth-functional operator.

DEFINITION – LITERAL WFF

A literal wff in **PL** is a well-formed formula in **PL** that consists of an atomic wff **P** or a negated atomic wff $\neg$**P**.

2.2.2 Scope

Increasingly complex formulas can be built up from the **PL** formation rules. Often, it will be convenient to talk about parts (or subformulas) of wffs.

DEFINITION – SUBFORMULA

A subformula **Q** of **P** is any wff occurring as a part of **P** (including **P** itself).

Example 1: Illustration of subformulas

- P, $\neg(P)$ are subformulas of $\neg(P)$
- P, Q, $(P \vee Q)$ are subformulas of $(P \vee Q)$
- $P, \neg(P), Q, (\neg(P) \vee Q))$ are subformulas of $(\neg(P) \vee Q)$
- $P, Q, P \to Q, \neg(P \to Q)$ are subformulas of $\neg(P \to Q)$

Some wffs are composed of multiple instances (or occurrences) of the same operator. For example, $\neg(\neg(P) \to Q)$ contains two instances of the $\neg$ operator. For convenience, we refer to each of the different instantiations of an operator as an "occurrence" of that operator.

The truth-functional operators of **PL** are said to have "scope".

DEFINITION – SCOPE OF **PL** OPERATOR

The scope of an occurrence of an operator in a wff **P** in **PL** is the smallest subformula of **P** that contains that occurrence of that operator.

Example 2: Illustration of the scope of operators

- P, $\neg(P)$ are subformulas of $\neg(P)$
  - the scope of $\neg$ is $\neg(P)$
- $P, Q, (P \vee Q)$ are subformulas of $(P \vee Q)$
  - the scope of $\vee$ is $(P \vee Q)$
- $P, \neg(P), Q, (\neg(P) \vee Q))$ are subformulas of $(\neg(P) \vee Q)$
  - the scope of $\neg$ is $\neg(P)$ while the scope of $\vee$ is $(\neg(P) \vee Q))$
- $P, Q, P \rightarrow Q, \neg(P \rightarrow Q)$ are subformulas of $\neg(P \rightarrow Q)$
  - the scope of $\rightarrow$ is $(P \rightarrow Q)$ while the scope of $\neg$ is $\neg((P \rightarrow Q))$

2.2.3 Main Operator

Some operators have more scope than others. The operator with the greatest or most scope is known as the "main operator".

Definition – main operator

The main operator of a **PL** wff is the truth-functional operator with the greatest scope.

Example 1: Illustration of the main operator of a wff

- the main operator of $\neg(P)$ is $\neg$
- the main operator of $(P \wedge Q)$ is $\wedge$
- the main operator of $(\neg(P) \vee Q)$ is $\vee$
- the main operator of $\neg((P \rightarrow Q))$ is $\neg$

In addition to the distinction between atomic and complex wffs, complex wffs can be further categorized in terms of their main operator (see Table 2.2).

1	A wff $\neg(\mathbf{P})$ is a **negated wff**
2	A wff $(\mathbf{P} \wedge \mathbf{R})$ is a **conjunction**
3	A wff $(\mathbf{P} \vee \mathbf{R})$ is a **disjunction**
4	A wff $(\mathbf{P} \rightarrow \mathbf{R})$ is a **conditional**
5	A wff $(\mathbf{P} \leftrightarrow \mathbf{R})$ is a **biconditional**

Table 2.2 – Five kinds of complex well-formed formulas

Some of the components (parts) of subformula have names (see Table 2.3).

The conjunction $(\mathbf{P} \wedge \mathbf{R})$ is composed of two *conjuncts* P and R.
The disjunction $(\mathbf{P} \vee \mathbf{R})$ is composed of two disjuncts P and R.
The conditional $(\mathbf{P} \rightarrow \mathbf{R})$ is composed of an antecedent P and a consequent R

Table 2.3 – Names for the components of conjunctions, disjunctions, and conditionals

The formation rules can be used to determine the main operator of a wff. To see this clearly, first notice that other than Rule 1, all of the formation rules are associated with a truth-functional operator, e.g. Rule 6 with '$\leftrightarrow$'. Second, the main operator of a wff is the truth-functional operator associated with the last formation rule applied to create the wff. For example, consider the use of the formation rules to show that $\neg(P \to \neg(R))$ is a wff:

1	Every propositional letter is a wff, so P and R are wffs.	Rule 1
2	If R is a wff, then $\neg(R)$ is a wff.	Line 1 + Rule 2
3	If P and $\neg R$ are wffs, then $P \to \neg(R)$ is a wff.	Lines 1, 2 + Rule 5
4	If $P \to \neg R$ is a wff, then $\neg(P \to \neg R)$ is a wff.	Line 3 + Rule 2

Notice that the last formation rule applied is Rule 2 (associated with '$\neg$ '). The '$\neg$ ' applied to '$P \to \neg(R)$' is the main operator.

Exercise 3: *Determine the main operator of the following wffs:*

1. $\neg(P)$
2. $\neg(\neg(P \wedge \neg(R)))$
3. $\neg(\neg P \leftrightarrow \neg R)$

2.2.4 Literal Negation

The literal negation of a proposition **P** is formed by placing parentheses around **P** and adding a negation to the resulting formula. In other words, it is the proposition that results from applying formation rule (2) on a proposition.

It is important to note that the literal negation of a proposition **P** is the negation of the entire formula and not just part of the formula.

	Proposition **P**	Literal negation of **P**
1	P	$\neg(P)$
2	$(P \to R)$	$\neg(P \to R)$
3	$(\neg(P) \wedge R)$	$\neg(\neg(P) \wedge R)$

Table 2.4 – Some Examples of the Literal Negation of **P**

Exercise 4: *Determine the literal negation of the following formulas, then state which one of the kinds of wffs that formula is (see* Table 2.2*):*

1. $\neg P$
2. $\neg P \wedge \neg R$

3. $\neg(\neg P \to \neg R)$

2.2.5 Three Conventions Concerning Parentheses

In order to make formulas more readable, three conventions are invoked. The first convention involves removing a set of parentheses if the scope of the negation operator is a single propositional letter or a negated propositional letter.

Proposition	Convention 1
$\neg(P) \wedge Q$	$\neg P \wedge Q$
$\neg(P)$	$\neg P$
$\neg(\neg(P))$	$\neg\neg P$

Table 2.5 – Convention 1

A second convention that is optionally employed involves substituting square brackets [ ] and curly brackets { } for additional sets of parentheses, namely for when there is a set of parentheses is contained in another set of parentheses.

Proposition	Convention 2
$(P \to Q) \vee ((S \vee R) \vee T)$	$(P \to Q) \vee [(S \vee R) \vee T]$
$\neg(P \wedge (Q \vee (R \to S)))$	$\neg\{P \wedge [Q \vee (R \to S)]\}$

Table 2.6 – Convention 2

A third convention is employed when a set of parentheses do not serve to disambiguate the scope of an operator. This occurs when there is a pair of open and closed parentheses and the next **PL** symbol to the immediate right and left of the open and closed parentheses is another parenthesis.

Proposition	Convention 3
$\neg((P \to Q))$	$\neg(P \to Q)$

Table 2.7 – Convention 3

2.3 PL: Semantics

PL is a set of symbols and a set of syntax (formation) rules for putting those symbols together. As such, the symbols and the wffs that are the result of applying the formation rules are *meaningless*. **PL** gets its meaning by being "interpreted". This section explains what an interpretation is and formulates a set of rules that allow for the assignment of truth values to wffs given an interpretation. However, before the notion of an interpretation is introduced, we need to introduce the notion of a function.

2.3.1 Function

What is a function? A function is a specific kind of relationship between two groups of things. The first group of things, known as the "domain", is composed of objects known as the function's "inputs" or "arguments". The second group of things, known as the "range", is composed of objects known as the function's "outputs" or "values". The relation between the inputs and the outputs is such that each input is related to one and only one output.

> DEFINITION – FUNCTION
>
> A function is a relation between two sets of things (the inputs and the outputs) such that each input is related to one and only one output.

Consider, for example, the following function which we will call the "lightening-color function". This function takes different a single input color and relates it to a lighter version of that color as an output color. In other words, it takes a single color as an argument and relates it to a color as a value.

Lightening Color Function: given a color as input, generate a lighter color as output.

So, for example, the lightening-color function takes brown and generates light brown.

Input		Output
grey	⇒	light grey
brown	⇒	light brown

Table 2.8 – Lightening Color Function

A function need not only involve one input. It may, for example, have two, or three, or any number of inputs. For example, consider a function that takes two colored items (red or blue) as input and produces an emotion (happy or sad) as output:

Color-Emotion Function: If the color-value input of both of the colored items is blue, then the output is happy. If the color-value input of either of the colored items is red, then the output is sad.

Input (color)	Input (color)		Output (emotion)
blue	blue	⇒	happy
blue	red	⇒	sad
red	blue	⇒	sad
red	red	⇒	sad

Table 2.9 – Color-to-Emotion Function

Suppose that someone's emotions are determined according to the color-emotion function as represented in Table 2.9 and suppose that we present them with two different pieces of clothing, a blue shirt and a blue pair of pants. The color-emotion function says that if the color-value input of both items is blue, then this individual will be happy. Alternatively, if we hand them a pair of blue pants and a red shirt, the color emotion function says that the individual will be sad.

The output of the function is determined not by anything outside (or unspecified by) the function. That is, a person's happiness or sadness (the values) are determined wholly by the input.

Note that while a function may relate a number of inputs to the same output, a function never relates a single input to more than one output. Consider a function that takes two colored and patterned rectangles as input and produces a single rectangle, which is a combination of both rectangles as output. Since there are an infinite number of color-shape combinations, an exhaustive set distinct diagrams is not possible, but the examples in Figure 2.2 and Figure 2.3 serve as useful illustrations of this function at work.

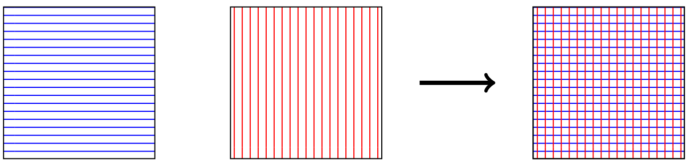

Figure 2.2 – Example 1 of the Rectangle-Pattern Function

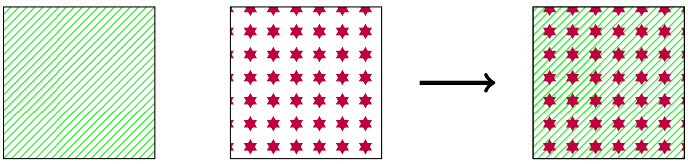

Figure 2.3 – Example 2 of the Rectangle-Pattern Function

Note that the function relates two different inputs to one and only one output.

2.3.2 Interpretation and Valuation

Now that we have the notion of a function in hand, we will consider two functions that pertain to the semantics of **PL**. Recall that the formal language of **PL** consists of a set of meaningless symbols and rules for putting these symbols together in the right way. Two functions supply meaning to **PL**: one takes propositional letters as input and assigns truth values to these propositional letters, the other assigns a truth value (T or F) to a wff depending put on the truth values of propositional letters and the operators that compose that wff.

> **Definition – Interpretation of PL**
>
> An interpretation of **PL** is a function that takes propositional letters as input and assigns them a single truth value (T or F) as output.

In other words, an interpretation assigns propositional letters meaning by assigning them one and only one truth value (T or F).

1. To symbolize the interpretation function of some proposition P, we write $\mathscr{I}(P) = T$ or $\mathscr{I}(Q) = F$, which reads "P is true under interpretation $\mathscr{I}$" or "Q is false under interpretation $\mathscr{I}$"

2. a single propositional letter can be interpreted in two different ways. Under one interpretation, it can be assigned a value of "T" while under a different interpretation it can be assigned a value of "F". In other words, if P is a propositional letter, P can be interpreted as true, i.e., $\mathscr{I}_1(P) = T$ or P can be interpreted as false, i.e., $\mathscr{I}_2(P) = F$.

3. an interpretation is an assignment of a truth value to every propositional letter in **PL**. While **PL** has an infinite number of propositional letters, in practice interpretations are specified by assigning truth values to the propositional letters that are under analysis, e.g. $\mathscr{I}(P) = T, \mathscr{I}(Q) = T, \mathscr{I}(R) = T, \mathscr{I}(Z) = T$

4. for n distinct propositional letters, there are 2^n interpretations. Thus for 1 propositional letter, $2^1 = 2$; for 2 propositional letters, $2^2 = 4$, and so on.

A second function is used to determine the truth values of all of the wffs of **PL**. This function, called a "valuation", assigns a truth value to a wff on the basis of the truth values assigned to propositional letters and the operators that compose the wff.

DEFINITION – VALUATION

For any interpretation $\mathscr{I}$, a valuation (v or $\mathcal{V}$) of a wff in **PL** is a function that assigns one and only one truth value (T or F) to each wff in **PL** in such a way that (let **P** and **Q** be variables for a wff in **PL**, **R** be a variable for a propositional letter in **PL**, and "iff" be an abbreviation for "if and only if"):

1. $v(R) = \mathscr{I}(R)$
2. $v\neg(P) = T$ iff $v(P) = F$
3. $v(P \wedge Q) = T$ iff $v(P) = T$ and $v(Q) = T$
4. $v(P \vee Q) = T$ iff $v(P) = T$ or $v(Q) = T$
5. $v(P \rightarrow Q) = T$ iff $v(P) = F$ or $v(Q) = T$
6. $v(P \leftrightarrow Q) = T$ iff $v(P) = T$ and $v(Q) = T$, or $v(P) = F$ and $v(Q) = F$

So if the interpretation function assigns truth values for all of the propositional letters, the valuation does the rest of the work to assign a truth values to wff by using the truth values of the propositional letters and the operators in the wff.

2.3.3 Truth Table

Our presentation of the meaning of the symbols and wffs in **PL** involved specifying two functions: the interpretation and the valuation function. A second,

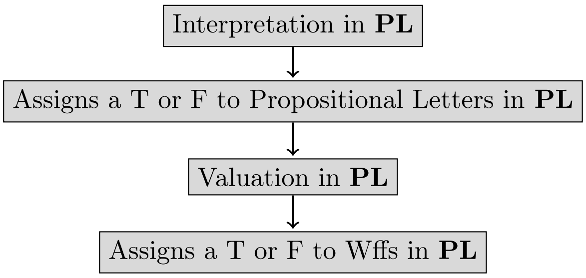

Figure 2.4 – An interpretation in **PL** is an assignment of truth values (T or F) to propositional letters; a valuation assigns truth values to wffs in **PL**.

and equivalent, method for presenting the meaning of the symbols and wffs in **PL** makes use of "truth tables". A truth table is simply a graphical display of the interpretation and valuation functions.

Truth Functions

First, consider that any propositional letter **P** can be interpreted in two ways. On one interpretation **P** can be assigned a value of T, while on another interpretation, it can be assigned a value of F. We can display these two interpretations of **P** as Truth Table 2.1.

	P
$\mathscr{I}_1$	T
$\mathscr{I}_2$	F

Truth Table 2.1 – Two interpretations of a propositional letter P

A truth table involving more than one propositional letter (e.g. P and Q) would take into account all of the different ways that P and Q can be interpreted (see Truth Table 2.2).

	P	Q
$\mathscr{I}_1$	T	T
$\mathscr{I}_2$	T	F
$\mathscr{I}_3$	F	T
$\mathscr{I}_4$	F	T

Truth Table 2.2 – Four interpretations of two propositional letters: P and Q

A truth table can be used to display the valuation function.

Negation

In the case of negated wffs $\neg P$: $\neg P$ is true if and only if P is false; $\neg P$ is false if and only if P is true.

P	$\neg P$
T	F
F	T

Truth Table 2.3 – Truth Table: Negation

Conjunctions

In the case of conjunctions $P \wedge R$: $P \wedge R$ is true if and only if P is true and R is true; $P \wedge R$ is false in all other cases.

P	R	$P \wedge R$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table 2.4 – Truth Table: Conjunction

Disjunctions

In the case of disjunctions $P \vee R$: $P \vee R$ is false if and only if P is false and R is false; $P \vee R$ is true in all other cases.

P	R	$P \vee R$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table 2.5 – Truth Table: Disjunctions

Conditionals

In the case of conditionals, a conditional $P \to R$ is true if and only if P is false or R is true.

P	R	$P \to R$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table 2.6 – Truth Table: Conditionals

Biconditionals

In the case of biconditionals, a biconditional $P \leftrightarrow R$ is true if and only if either P and R are true or P and R are false.

P	R	$P \leftrightarrow R$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table 2.7 – Truth Table: Biconditionals

Exercise 5: *For each of the following propositions, determine if they are true or false given the interpretation provided.*

1. $\neg(P)$ if $\mathscr{I}(P) = F$
2. $P \to Q$ if $\mathscr{I}(P) = F, \mathscr{I}(Q) = F$
3. $P \wedge Q$ if $\mathscr{I}(P) = T, \mathscr{I}(Q) = T$
4. $P \vee Q$ if $\mathscr{I}(P) = F, \mathscr{I}(Q) = T$
5. $P \leftrightarrow Q$ if $\mathscr{I}(P) = T, \mathscr{I}(Q) = T$

2.4 PL: Translation

We have at our disposal two languages: an abstract truth-functional language (**PL**) and a natural language (English). One way that these two language relate to each other is that a number of English propositions and arguments can be expressed by **PL** wffs.

Single propositional letters (atomic propositions) in **PL** can be used to capture simple English sentences that express propositions that do not contain the use of truth-functional uses of expressions like 'not', 'and', 'or', etc. (this will become a little clear as you consider the remaining truth-functional operators).

John is kind.	J
John will go to heaven	H
John went to the store.	S

2.4.1 Negation

Negated propositions $\neg\mathbf{P}$ in **PL** capture English uses of ‘not’, ‘it is not the case that’, ‘it is false that’.

John is not kind.	$\neg J$
It is not the case that John will go to heaven.	$\neg H$
John did not go to the store.	$\neg S$

For consider that a proposition John is not kind is true just in the case that John is kind is false and John is not kind is false just in the case that John is kind is true.

2.4.2 Conjunctions

Conjunctions $\mathbf{P} \wedge \mathbf{Q}$ in **PL** capture truth-functional uses of ‘and’ in English.

John is kind and John will go to heaven.	$J \wedge H$
John is kind and John will go to the store.	$J \wedge S$
John is kind, John will go to heaven, and John went to the store.	$(J \wedge H) \wedge S$

For consider that a proposition John is kind and John will go to heaven is true just in the case that John is kind is true and John will go to heaven is true. It is false if either of those propositions (or both) are false.

2.4.3 Disjunctions

Disjunctions $\mathbf{P} \vee \mathbf{Q}$ in **PL** capture the inclusive truth-functional use of ‘or’ in English:

John is kind or John will go to heaven.	$J \vee H$
John is kind or John went to the store.	$J \vee S$
John is not kind or John will not go to heaven.	$\neg J \vee \neg H$

For consider that a proposition John is kind or John will go to heaven is false if and only if John is kind is false *or* John will go to heaven is false. In all other cases, this proposition is true. That is:

1. If John is kind is true, then John is kind or John will go to heaven is true.
2. If John will go to heaven is true, then John is kind or John will go to heaven is true.
3. If John is kind is true AND John will go to heaven is true, then John is kind or John will go to heaven is true.

2.4.4 Material Conditionals

Conditionals are claims of the form “if ..., then ...”. $\mathbf{P} \rightarrow \mathbf{Q}$ in **PL** expresses one species of conditional known as the “material conditional”. A material conditional is a conditional whose truth value is determined by the valuation function represented in Truth Table 2.6. Some examples::

If John is kind, then John will go to heaven.	$J \to H$
If John is kind, then John went to the store.	$J \to S$
If John is in Toronto, then John is in Canada	$J \to C$

Not every conditional is a material conditional (see Table 2.10). That is, some conditionals are not material conditionals in that their truth or falsity is not the result of a truth function.

Definitional	If John is a hypochondriac, then he is a person who is abnormally anxious about his health.
Causal	If you pull the trigger, the gun will fire.
Decisional	If you go to the party, then I will go to the party.

Table 2.10 – Three Kinds of Conditionals

Definitional, causal, and decisional conditionals are not material conditionals in that their truth depends, not merely on the truth or falsity of the antecedent and consequent. The truth values of these conditionals also depends upon additional factors concerning the relationship between the antecedent and consequent of the conditional. For example, the causal conditional is said to express (i) that on the condition that you pull the trigger, the gun will fire, but also (ii) that pulling the trigger *causes* the gun to fire. In other words, this conditional can be false when both the antecedent and the consequent are true, namely in the case where you pull the trigger and the gun fires, but where your pulling of the trigger does not cause the gun to fire.

Insofar as no additional factors concerning the relationship between the antecedent and consequent plays a role in the truth (falsity) of the material conditional, the material conditional expresses something weaker than the definitional, causal, and decisional conditionals. That is, the truth or falsity of the material conditional is wholly determined by the truth and falsity of the antecedent and consequent of the conditional. No additional factors concerning whether the antecedent causes the consequent is considered.

Paradoxes of Material Implication

In allowing the truth of material conditionals to be wholly determined by the truth of the antecedent and consequent, two seemingly counter-intuitive results emerge:

- the conditional is true when the antecedent is false (independent of the truth/falsity of the consequent)
- the conditional is true when the consequent is true (independent of the truth/falsity of the antecedent)

These results are known as the “paradoxes of material implication”. An example of the first result is “If rain is made of spaghetti sauce, then God exists.” According to our valuation of the material conditional, since rain is not made of spaghetti sauce, the antecedent is false, but the conditional is true. It is true even if irrespective of the the truth of the consequent. An example of the second result is "If God exists, then there are dandelions on earth." According to our valuation of the material conditional, since there are dandelions on earth, the consequent is true, and so is the conditional, irrespective of the truth of the antecedent.

The above two results suggest that our valuation of a conditional $P \to Q$ as true when either $v(Q) = T$ or when $v(P) = F$ is incorrect.

Saving our Intuitions I: Rows 3 and 4 are not false

Consider the following example:

(1) If John is in Toronto, then John has five dollars in his pocket.

(1) expresses something about John. Namely, it says that in the case that John is in Toronto, he has five dollars in his pocket. Intuitively, (1) is true if John is in Toronto and he has five dollars in his pocket. In addition, it is intuitively the case that (1) is false if John is in Toronto but he does not have five dollars in his pocket. What about cases where John is not in Toronto? If (1) is thought to express the modest claim that *on the condition that John is in Toronto, then he has five dollars in his pocket*, (1) is **not false** in cases where John is not in Toronto for (1) does not say where John is located.

Saving our Intuitions II: Unbroken Promises

One way of explaining why $v(P \to Q) = T$ when $v(P) = F$ is given by Mark Nelson who contends that "false conditionals are like broken promises and true conditionals are like unbroken promises" (1993:155). To illustrate the connection between promises and conditionals, let's begin by supposing that God exists and he issues you the following promise:

> If you are a kind, you will go to heaven.

Let's abbreviate "you are kind" as P and "you will go to heaven" as R.

$$P \to R$$

Now let's consider the conditions under which God's promise is true and the conditions under which his promise is false. First, suppose that all of your life, you are very kind. You help your neighbors, you give to the poor, and you smile at everyone you see. Thus, $v(P) = T$. Now, when you die, you go to heaven. Thus, $v(R) = T$. We said that a false conditional is like a broken promise and a true conditional is like an unbroken promise. So, has God broken his promise? No, God kept his promise.

Next, suppose that all of your life, you are very kind. You help your neighbors,

P	R	$P \to R$
T	T	T
T	F	
F	T	
F	F	

Truth Table 2.8 – Truth Table: Conditionals

give to the poor, and smile at everyone you see. Thus, $v(P) = T$. But, when you die, you do not go to heaven. Instead, you are thrust into the hottest fires in hell. Thus, $v(R) = F$. Has God kept his promise? No, God promised that if you were kind, he would send you to heaven, but he didn't send you to heaven even though you were kind. We said that a false conditional is like a broken promise and a true conditional is like an unbroken promise. In this case, since God has broken his promise, we say that when P is true, but R is false, then $P \to R$ is false.

P	R	$P \to R$
T	T	T
T	F	F
F	T	
F	F	

Truth Table 2.9 – Truth Table: Conditionals

Let's consider a third case. Suppose that you are not kind. You make life harder for your neighbors, you steal from the poor, and you scowl at everyone you see. But, when you die, God sees what a pitiful wretch you are and decides to send you to heaven anyway. Has God, in sending you to heaven, broken his promise? The answer is no. God did not assert that he would send you to heaven if *and only if* you were kind. Rather, God's promised that if you are kind, he will send you to heaven, but God did not say what he would do if he weren't kind. And so, if a false conditional is like a broken promise and a true conditional is like an unbroken promise, and God has not broken his promise, then $v(P \to R) = T$.

P	R	$P \to R$
T	T	T
T	F	F
F	T	T
F	F	

Truth Table 2.10 – Truth Table: Conditionals

Finally, suppose that you are not kind. Again, you are a cruel person, diabolical to the core. And, when you die, God sees what a pitiful wretch you are and decides not to send you to heaven. Instead, he throws you into the darkest corners of hell. Has God, in sending you to hell, broken his promise? The answer is no. God's bargain with you was that if you are kind, then he will send you to heaven, but you were not kind, so God is under no obligation to send you to heaven.

P	R	$P \rightarrow R$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table 2.11 – Truth Table: Conditionals

2.4.5 Biconditional

Biconditionals $\mathbf{P} \leftrightarrow \mathbf{Q}$ in **PL** capture the truth-functional use of 'P if and only if Q' propositions in English:

John is kind if and only if John will go to heaven.	$J \leftrightarrow H$
John is kind if and only if John went to the store.	$J \leftrightarrow S$
John is in Toronto if and only if John is in Canada	$J \leftrightarrow C$

Exercise 6: *Translate the following propositions into PL*

1. John is a zombie.
2. John is a zombie and Mary is happy.
3. If John is a zombie, then Mary is happy.
4. If John is not a zombie, then Mary is not happy.
5. Either John is a zombie or Mary is happy.
6. John is a zombie if and only if Mary is happy.
7. If John is a zombie and Mary is happy, then either John is a zombie or Mary is not happy.
8. It is not the case that John is not a zombie.

2.5 PL: More Complex Translations

In this section, more complex translation sentences in English are translated into **PL**.

- P and Q and R
- P or Q or R
- neither P nor Q
- not both P and Q
- P only if Q
- P even if Q

First, sets of propositions connected by the truth-functional use of "and" or "or" can all be translated using multiple instances of $\wedge$ and $\vee$ respectively. Often, multiple instances of "and" are replaced by the use of commas.

Example 1: P and Q and R, P or Q or R

John is tall and Mary is happy and Frank is sweet.	$(J \wedge M) \wedge F$
John is tall or Mary is happy or Frank is sweet.	$(J \vee M) \vee F$
John is tall, Mary is happy, and Frank is sweet.	$(J \wedge M) \wedge F$

Some propositions have the following structure: "neither P nor Q". These statements are true just in the case that P is false and Q is false. Alternatively put, these statements are truth just in the case that not-P is true and not-Q is true.

Example 2: neither P nor Q

John is neither happy nor angry.	$\neg H \wedge \neg A$
The stock market will neither go up nor down today.	$\neg U \wedge \neg D$

Another set of propositions have the following structure: "not both P and Q". Propositions of this type do not express the strong claim that not-P and not-Q is the case but rather that it is not the case that *both* P and Q are true. Considering that $P \wedge Q$ is true just in the case that both P and Q are true, we translate statements of the form "not both P and Q" as $\neg(P \wedge Q)$.

Example 3: not both P and Q

John is not both happy and angry.	$\neg(H \wedge A)$
John is happy or angry, but not both.	$(H \vee A) \wedge \neg(H \wedge A)$

Propositions of the form "P only if Q" express that P is not the case if Q is not the case. In other words, propositions of the form "P only if Q" simply state "not-P if not-Q", which can be expressed as "if not-Q, then not-P"

Example 4: P only if Q

John will go to the party only if Mary goes.	$\neg M \to \neg J$
John will be found guilty only if he committed the crime.	$\neg C \to \neg G$

Propositions of the form "P even if Q" express the fact that P is the case regardless of the truth or falsity of Q. For this reason, we can represent "P even if Q as simply asserting P

Example 5: P even if Q

John will go to the party even if Mary goes.	P
The stock market will go up even if no one buys stock.	P

Exercise 7: *Translate the following English propositions into* **PL**

1. John is happy, Mary is tired, and Frank is surprised.
2. John is happy or Mary is not tired or Frank is not surprised.
3. Neither John is happy nor Mary is tired.
4. Either John is happy or Mary is tired, but not both.
5. John knows how to dance only if pigs know how to fly.
6. I will be at the party even if the party is cancelled.

Questions

1. List all of the symbols of **PL**
2. How many propositional letters are there in **PL**?
3. Is the following a wff in **PL**: $(\neg(P) \to \neg(Q))$?
4. Use the **PL** formation rules to prove that $(\neg(P) \vee \neg(Q))$ is a wff in **PL**.
5. What is the difference between an atomic wff and a complex wff?
6. What is the scope of the leftmost occurrence of $\neg$ in $(\neg(P) \vee \neg(Q))$?
7. What are all of the subformulas of $(\neg(P) \vee \neg(Q))$?
8. What is the main operator of a wff?
9. What is the main operator of $(\neg(P) \vee \neg(Q))$?
10. What is the literal negation of $(\neg(P) \vee \neg(Q))$?
11. How would you represent the following wff using the conventions for making wffs more readable: $(((\neg P) \vee Q) \wedge R)$
12. What is a function?
13. Can the input (item in a domain) of a function be related to more than one output (item in the range)? Why or why not?
14. What would be wrong with saying that there is a function that relates letters A, B, C (the inputs) to numbers 1, 2, 3 (the outputs) whereby A is related to 2, B is related to 1, and C is related to 2?
15. What is an interpretation in **PL**?
16. Can a single propositional letter P be assigned a truth value of both T and F under a single interpretation?
17. If a single propositional letter P receives an interpretation, is there anything wrong in saying that P is neither true nor false?
18. What is a valuation in **PL**?
19. What is the difference between a valuation and an interpretation?
20. Using the valuation function, what is the truth value for $P \to Q$ if $v(P) = T$ and $v(Q) = F$?
21. How would you translate the following English sentences in **PL**:
    (a) It is not the case that John is happy.
    (b) If John is happy, then Mary is happy.
    (c) John is happy and Mary is happy.
    (d) John is happy or Mary is smart.
    (e) John is happy if and only if Mary is happy.