Predicate Derivations: Four Quantifier Rules

The proof system for RL is RD, which consists of PD+ (the proof system from propositional logic) plus additional derivation rules for quantified formulas.

Universal Elimination (∀E)

	Universal Elimination (∀E) From any universally quantified proposition (∀x)**P**, we can derive a substitution instance **P**(a/x) in which all bound variables are consistently replaced with any individual constant (name).		(∀x)**P** **P**(a/x)	∀E

The idea is that if you have a universally quantified proposition (∀x)**P**, you can move forward a step in the proof with a proposition **P**(a/x) that is the result of removing the universal quantifier and replacing any bound variables with any name of your choosing.

Example #1

1	(∀x)Px	P
2	Pa	1∀E
3	Pb	1∀E
4	Pe	1∀E

Replacement must be *uniform*.

1	(∀x)Pxx	P
2	Paa	1∀E – **OK!**
3	Pab	1∀E – **NO!**
4	Pba	1∀E – **NO!**

Existential Introduction (∃I)

	Existential Introduction (∃I) From any possible substitution instance **P**(a/x), an existentially quantified proposition (∃x)**P** can be derived by consistently replacing at least one individual constant (name) with an existentially quantified variable.		**P**(a/x) (∃x)**P**	∃I

The idea is that if you have a non-quantified proposition **Pa**, you can move forward a step in the proof with an existentially quantified proposition (∃x)**P**(x/a) that is the result of uniformly replacing at least one name with an existentially quantified bound variable.

Example #1

1	Pa	P
2	Rbb	P
3	(∃x)Px	1∃I
4	(∃x)Rxx	2∃I

Replacement must be *uniform* but you need only replace *one name*.

1	Lab	P
2	Rbb	P
3	(∃x)Lax	1∃I – **OK!**
4	(∃x)Lxa	1∃I – **OK!**
5	(∃x)Lxx	1∃I – **NO!**
6	(∃x)Rxx	2∃I – **OK!**
7	(∃x)Rbx	2∃I – **OK!**

Practice

1. Paa→Rbb, Pbb→Rcc, (∀x)Pxx ⊦ Rbb∧Rcc
2. (∃x)Rx→(∃x)Mx, Ra ⊦ (∃x)Mx
3. Pab, (∃x)(∃y)Pxy→(∀x)Zxx ⊦ Zaa
4. (∀x)(∀y)Pxy, Pab→Rc, Pde→Rc ⊦ (∃x)(Rx∧Rx)
5. (∀x)(∀y)Pxy, Paa→Lab ⊦ (∃y)(∃x)Lxy

6.* Labc, (∃x)(∃y)(∃z)Lxyz→¬(∀x)Pxxx, (∀x)Pxxx∨(∀x)(∀y)(∀z)Mxyz ⊦ Mdab

Universal Introduction

	Universal Introduction (∀I) A universally quantified proposition (∀x)**P** can be derived from a possible substitution instance **P**(a/x) provided (1) *a* does not occur as a premise or as an assumption in an open subproof, and (2) *a* does not occur in (∀x)**P**.		**P**(a/x) (∀x)**P**	∀I

The idea is that if you have a non-quantified proposition **Pa**, you can move forward a step in the proof with a universally quantified proposition (∀x)**P**(x/a) that is the result of uniformly replacing each name with universally quantified bound variables. However, the names cannot occur in the premises, as the assumption in an open subproof, and cannot occur in the universally quantified proposition (∀x)**P**(x/a) derived.

Example #1

1	(∀x)Pxx	P
2	Paa	1∀E
3	(∀y)Pyy	2∀I

Example #2

1	│ Raa	A
2	│ Raa	1R
3	Raa→Raa	1-2→I
4	(∀x)(Rxx→Rxx)	3∀I

In the above example, (∀I) is applied to line 3 even though ‘a’ is in the assumption. However, the subproof involving ‘a’ is no longer open.

First Violation of Restriction #1

1	Paa	P
2	(∀x)Pxx	1∀E — **NO!**

Second Violation of Restriction #1

1	│ Raa	A
2	│ (∀x)Rxx	1∀I — **NO!**

Violation of Restriction #2

1	(∀x)Pxx	P
2	Paa	1∀E
3	(∀y)Pya	2∀I — **NO!**

Existential Elimination

	Existential Elimination (∃E) From an existentially quantified expression (∃x)**P**, an expression **Q** can be derived from the derivation of an assumed substitution instance **P**(a/x) of (∃x)**P** provided (1) the individuating constant *a* does *not* occur in *any* premise or in an active proof (or subproof) *prior* to its arbitrary introduction in the assumption **P**(a/x) and (2) the		(∃x)**P** │ **P**(a/x) │ . │ . │ . │ **Q** **Q**	∃E

	individuating constant *a* does not occur in proposition **Q** discharged from the subproof.			

The idea is that if you have an existentially quantified proposition (∃x)**P**, you can derive **Q** provided you assume **P**(a/x) then derive **Q** with that subproof . However, (1) 'a' in **P**(a/x) cannot occur in the premise or an active proof and (2) 'a' from **P**(a/x) does not occur in **Q**.

Example #1

1	(∃x)Px	P
2	│ Pa	A / ∃E
3	│ (∃y)Py	2∃I
4	(∃y)Py	1, 2-3∃E

First Violation of Restriction #1

1	(∃z)Wzz	P
2	Wbb→Lc	P
3	│ Wbb	A / ∃E
4	│ Lc	2,3→E
5	Lc	1,3-4∃I— **NO!**

Second Violation of Restriction #1

1	(∀z)(Pzz→Lzz)	P
2	(∃z)Pzz	P
3	│ Pbb→Wcc	A
4	│ │ Pbb	A / ∃E
5	│ │ Wcc	3,4→E
6	│ Wcc	2, 4-5∃E — **NO!**

Violation of Restriction #2

1	(∃z)Wzz	P
2	│ Wbb	A / ∃E
3	│ (∃x)Wbx	2∃I
4	(∃x)Wbx	1, 2-3 ∃E — **NO!**

Practice

1. (∀x)(∀y)Lxy, (∀x)Lxx→(∃x)(∃y)Lxy ⊢ (∃x)(∃y)Lxy
2. (∃x)Pxx ⊢ (∃x)(Pxx∨Rx)
3. (∃x)(∃y)Pxy ⊢ (∃y)(∃z)Pyz
4. (∀x)(Pxx→Pxx)→Sa ⊢ Sa
5. (∀x)(Pxx→Pxx)→(∃x)Mx ⊢ (∃x)(Mx∧Mx)
6. (∃x)(∃y)(∃z)(Lxy∧Lyz), (∃x)(∃y)Lxy→(∀z)(∀y)Pzy ⊢ (∀x)Pxx

Predicate Derivations: Quantifier Negation

In addition to the four introduction and elimination rules for quantified propositions, we can add to RD an equivalence rule (or rule of replacement). This rule, known as Quantifier Negation, allows us to replace negated quantified subformulas with non-negated quantified subformulas, and vice versa.

	Quantifier Negation (QN) From a negated universally quantified expression $\neg(\forall x)\mathbf{P}$, an existentially quantified expression $(\exists x)\neg\mathbf{P}$ can be derived, and vice versa. Also, from a negated existentially quantified expression $\neg(\exists x)\mathbf{P}$, a universally quantified expression $(\forall x)\neg\mathbf{P}$ can be inferred, and vice versa.		$\neg(\forall x)\mathbf{P} \dashv\vdash (\exists x)\neg\mathbf{P}$ $\neg(\exists x)\mathbf{P} \dashv\vdash (\forall x)\neg\mathbf{P}$	QN QN

Example 1

1	$\neg(\forall x)Px$	P
2	$(\exists x)\neg Px$	1QN
3	$\neg(\forall x)\, Px$	2QN

Remember, since (QN) is a rule of replacement, it can be applied to subformula

1	$\neg\, (\forall x)Px\rightarrow Pa$	P
2	$(\exists x)\neg Px\rightarrow Pa$	1QN

Practice
1. $(\forall x)Pxx\rightarrow(\exists x)Rx,\ (\forall x)\neg Rx\ \vdash (\exists x)\neg Pxx$
2. $(\forall x)\neg(\exists y)Pxy\rightarrow(\forall x)Zx,\ (\forall x)(\forall y)\neg Pxy\ \vdash (\forall x)Zx$
3. $\neg(\exists x)(\exists y)Pxy,\ (\forall x)(\forall y)\neg Pxy\rightarrow(\forall z)Zz\ \vdash (\forall z)Zz$
4. $\neg(\forall x)Px\ \vdash (\exists x)(\neg Px\vee Zx)$
5. $\neg(\forall x)(\forall z)Pxz,\ (\exists x)(\exists z)\neg Pxz\rightarrow(\forall z)Lz \vdash (\forall w)(Lw\vee Mw)$
6. $\vdash (\exists x)Px\rightarrow(\forall x)(Px\rightarrow Px)$