Handout 5

PL Derivations

In this chapter, we specify a deductive apparatus for **PL**.

> DEFINITION – DEDUCTIVE APPARATUS
>
> A deductive apparatus for **PL** is a set of rules of inference (or "derivation" rules) that determine which ways that formulas can be transformed. That is, it is a list of permissible rules that express which wffs **Q** can be written after which formulas **P** in a proof. The deductive apparatus for **PL** is hereafter abbreviated as **PD**.

We can think of the deductive apparatus as motivated by arguing in everyday life. Suppose two people Tek and Liz agree on a great many things. They have similar life experiences, they read many of the same scientific studies, and they have similar values. Let's refer to the set of propositions that Tek and Liz both take as true Γ (where Γ just represents a set of propositions, e.g. $A, B, C, \ldots M$). Now suppose that Tek thinks that from Γ, we can easily reason to another set of propositions P, Q, R. Tek contends that if we believe Γ then we ought to also believe P, Q, R. In contrast, Liz says that even if all of the propositions in Γ are true, there is no way to reason to P, Q, R.

Tek and Liz don't disagree about any facts concerning the world. What they disagree about is whether P, Q, R follows from Γ, or what follows from what. To fix this problem, they decide to develop a set of rules that specify how one can reason from one proposition (or groups of propositions) to another. The rules are formulated in a highly general (abstract) way so that it can apply to any particular subject matter. This set of rules is their deductive apparatus.

> DEFINITION – DERIVATION OF **Q** IN **PD**
>
> A derivation in **PD** of **Q** is a finite (not infinite and not empty) string of formulas from a set Γ of **PL** wffs where (i) the last formula in the string is Q and (ii) each formula is either a premise, an assumption, or is the

result of the preceding formulas and the deductive apparatus.

Definition – syntactic consequence

A formula **Q** is a syntactic consequence in **PD** of a set Γ of **PL** wffs if and only if there is a derivation in **PD** of Q from Γ. To express that **Q** is a syntactic consequence of Γ, we write $\Gamma \vdash Q$.

We can think of a derivation using **PD** and the notion of syntactic consequence as the result of Tek and Liz using their deductive apparatus to reason to some proposition. For instance, suppose that Tek and Liz agree upon a set of rules that permit reasoning from one proposition to the next. Tek then contends that if Tek and Liz both accept Γ then they also should accept Q. Liz is unconvinced. To convince her of Q, Tek writes down all of the propositions expressed by Γ and then shows how, using the deductive apparatus, Q can be written down as well. This process of writing down premises, making assumptions, and propositions permitted by the deductive apparatus is a "derivation". Saying that Q is a syntactic consequence of Γ merely expresses the fact that there is a derivation from Γ to Q

5.1 Setup

Consider the following: $P \wedge R, Y \rightarrow R \vdash Z$. This set of symbols says that Z is a syntactic consequence of $P \wedge R, Y \rightarrow R$. As such, it says that there is a derivation from $P \wedge R, Y \rightarrow R$ to Z. How do we show that there is such a derivation. First, a derivation begins with an initial setup involving three columns:

1. for numbering the premises,
2. writing (stacking) the propositions,
3. justification of propositions and indicating the goal proposition (or conclusion)

1	$P \wedge R$	P
2	$Y \rightarrow R$	P, Z

In the setup of the above derivation, '$P \wedge R$' and '$Y \rightarrow R$' are premises (and we use 'P') to indicate this. The conclusion (goal proposition) is 'Z'.

5.2 Proofs: Intelim Derivation Rules

In what follows, we develop the deductive apparatus (**PD**). The particular type of deductive apparatus developed here is known as a system of "natural deduction" as the particular rules are akin to certain rules of inference (or reason) people use in everyday arguments. The particular rules of **PD** will be called the "derivation rules". There are two main types of derivation rules: **introduction rules** (these introduce a proposition of a certain type into a proof) and **elimination rules** (these begin from propositions of a certain type in a proof).

> DERIVATION RULE – CONJUNCTION INTRODUCTION $\wedge I$
>
> From P and Q, we can derive $P \wedge Q$. Also, from P and Q, we can derive $Q \wedge P$.
>
> $P, Q \vdash P \wedge Q$
> $P, Q \vdash Q \wedge P$

We represent derivation rules in a variety of different ways. First, we can represent them as tree-like structures where propositions below a line represent those that are "deduced" or "derived" from those above the line. Second, we can provide a minimal working example of the derivation rule (see Figure 5.1). In this handout, we represent derivation rules using the latter method.

$$\frac{\text{P} \qquad \text{Q}}{P \wedge Q} \wedge I$$

1	P	P
2	Q	P
3	$P \wedge Q$	$\wedge I$, 1, 2

Figure 5.1 – Two Ways of Representing a Derivation. Where P and Q are metavariables for any wff in **PL**

Conjunction introduction states that from two different propositions, we can derive the conjunction of these propositions. For example, prove: $P, R, Z \vdash P \wedge Z$

1	P	P
2	R	P
3	Z	P
4	$P \wedge Z$	$\wedge I$, 1, 3

> DERIVATION RULE – CONJUNCTION ELIMINATION ($\wedge E$)
>
> From $P \wedge Q$, we can derive P. Also, from $P \wedge Q$, we can derive Q.
>
> $P \wedge Q \vdash P$ or $P \wedge Q \vdash Q$

Conjunction elimination states that from a conjunction, we can derive either of the conjuncts. For example, prove: $P \wedge Z \vdash Z$

1	$P \wedge Z$	P
2	Z	$\wedge E$, 1

Exercise 1:

1. $P \wedge (R \wedge M) \vdash R$
2. $P \wedge (R \wedge M) \vdash M$
3. $P, R, M \vdash (M \wedge P) \wedge R$
4. $P, R, M \vdash (M \wedge R) \wedge R$
5. $P \wedge R, \neg Z \wedge \neg W \vdash P \wedge \neg W$
6. $P \wedge (\neg R \wedge \neg W), L, R \wedge Z \vdash \neg R \wedge (L \wedge Z)$

5.2.1 Assumptions & Subproofs

An assumption (abbreviated as 'A') is a proposition taken to be, or assumed, true for the purpose of proof. Each time you make an assumption, you indent, draw a line indicating that you are moving into a subproof, and justify that proposition you assumed with an 'A'.

$$
\begin{array}{ll}
1 & MainLine \\
2 & \quad | \; \underline{Assumption} \\
3 & \quad | \; Subproof \\
4 & \quad | \; . \\
5 & \quad | \; . \\
6 & \quad | \; .
\end{array}
$$

After you've made an assumption, you can reason within the subproof that has been created by the assumption:

$$
\begin{array}{lll}
1 & S & \text{P} \\
2 & \quad | \; \underline{B} & \text{A} \\
3 & \quad | \; S \wedge B & \wedge I, \text{1, 2} \\
4 & \quad | \; . & \\
5 & \quad | \; . & \\
6 & \quad | \; . &
\end{array}
$$

The above is similar to saying 'Let's agree that 'S' is true. Now, let's assume 'B' is true. Well, if 'S' is true, then given our assumption 'B', it follows that '$S \wedge B$'.

You are not limited to one assumption. You can make assumptions within assumptions. For example, consider the proof just below. This proof reads something like the following:

Let's say that 'Q' is true. Given that 'Q' is true, let's assume 'S'. Now that we've assumed 'S', let's assume 'W'.

1	Q	P
2	| S	A
3	| | W	A
4	| | .	
5	| | .	
6	| | .	

You can think of subproofs like containers or nests. That is the subproof begun by S contains the subproof begun by W. Likewise, the mainline of the proof, beginning with 'Q' contains the subproofs begun by 'S' and 'W'. In the language of nests, 'W' is in the nest begun by 'S' and 'S' is in the nest of the main line of the proof. 'W' is in the most deeply nested part of the proof while 'Q' is in the least deeply nested part.

In addition, sometimes in proofs you will make an assumption and then make another assumption that is not related to the first assumption:

1	A	P
2	| B	A
3	| $A \wedge B$	$\wedge I$, 1, 2
4	| .	
5	| .	
6	| C	A
7	| $A \wedge C$	$\wedge I$, 1, 6
8	| .	
9	| .	

You can use derivation rules to reason within subproofs, but there are certain restrictions. The basic rule is the following:

If P is in a section of the proof S1 that contains another subproof S2, then P can be used in S2. If R is in a section of the proof S3 that does not contain a subproof S4, then R cannot be used in S4.

5.2.2 Assumptions & Subproofs: Violations!

The above rule is violated when using a proposition inside a subproof, you derive a proposition outside the subproof.

Example # 1: Z is in the subproof and used to derive $Z \wedge R$, a proposition which is not in the subproof containing Z

1	R	P
2	| Z	A
3	| $R \wedge Z$	$\wedge I$, 1, 2
4	$Z \wedge R$, *NO*!	$\wedge I$, 1, 2

Example # 2: B is in the subproof and used to derive $B \wedge C$, a proposition which is not in the subproof containing B

1	A	P
2	| B	A
3	| $A \wedge B$	$\wedge I$, 1, 2
4	| C	A
5	| $B \wedge C$, *NO*!	$\wedge I$, 2, 4

Derivation Rule – Conditional Introduction ($\rightarrow I$)

From a derivation of Q within a subproof involving an assumption P, we can derive $P \rightarrow Q$ out of the subproof.

n	| P	A
$\vdots$	| $\vdots$	
$(n+1)$	| Q	
$(n+2)$	$P \rightarrow Q$	$\rightarrow I$, n–$(n+1)$

Conditional introduction allows for introducing a conditional $P \rightarrow Q$ outside of a subproof given a subproof containing P as the assumption and Q as a derived proposition within that subproof.

Here is an example: $R \vdash Z \rightarrow R$

1	R	P
2	$\vert\ Z$	A
3	$\vert\ Z \wedge R$	$\wedge I$, 1, 2
4	$\vert\ R$	$\wedge E$, 3
5	$Z \to R$	$\to I$, 2–4

Exercise 2:

1. $R \wedge Z \vdash W \to Z$

DERIVATION RULE – CONDITIONAL ELIMINATION ($\to E$)

From $P \to Q$ and P, we can derive Q.

$P \to Q, P \vdash Q$

Conditional elimination allows for deriving a proposition Q provided we have a conditional $P \to Q$ and the antecedent P of that conditional.

Here is an example: $Z \to R, Z \wedge P \vdash R$

1	$Z \to R$	P
2	$Z \wedge P$	P R
3	Z	$\wedge E$, 2
4	R	$\to E$, 1, 3

Exercise 3:

1. $R \to Z, Z \to W, R \wedge M \vdash W \wedge M$

DERIVATION RULE – REITERATION (R)

From P we can derive P.

$P \vdash P$

Reiteration allows for deriving a proposition P provided P already occurs in the proof. Here are two examples.

Example # 1: $Z \vdash Z$

1	Z	P, Z
2	Z	R, 1

Example # 2: $R \vdash Z \rightarrow R$

1	R	$P, Z \rightarrow R$
2	$\quad\vert\; Z$	A
3	$\quad\vert\; R$	R, 1
4	$Z \rightarrow R$	$\rightarrow I$, 2–3

Exercise 4:

1. $P \rightarrow Z, P \vdash P \wedge Z$
2. $(P \wedge Z) \rightarrow W, P, Z \vdash W$
3. $P \vdash R \rightarrow P$
4. $P, M \vdash (R \vee F) \rightarrow (P \wedge M)$
5. $(R \vee F) \rightarrow Z, M \wedge (R \vee F) \vdash M \wedge Z$
6. $R \rightarrow P, R \wedge L, \vdash P \rightarrow (L \wedge R)$
7. $\vdash R \rightarrow R$
8. $\vdash R \rightarrow (R \wedge R)$
9. $\vdash R \rightarrow [Z \rightarrow (M \rightarrow Z)]$

A Word of Encouragement

For students taking a first course in symbolic logic, proofs tend to be one of the most difficult topics to grasp. Unlike decision procedures (truth tables and truth trees), the process of solving a proof requires the use of strategies and a little trial-and-error. As you work through various proofs, don't get discouraged if you are unable to get the answer immediately or if you have to start a proof over. Just keep at it and practice, practice, practice!

Derivation Rule – Negation Introduction ($\neg I$)

From a derivation of a proposition Q and its literal negation $\neg Q$ within a subproof involving an assumption P, we can derive $\neg P$ out of the subproof.

$$
\begin{array}{lll}
n & \quad\vert\ \underline{P} & \text{A} \\
\vdots & \quad\vert\ \vdots & \\
(n+1) & \quad\vert\ Q & \\
(n+2) & \quad\vert\ \neg Q & \\
(n+3) & \neg(P) & \neg I,\ n\text{–}(n+2)
\end{array}
$$

Derivation Rule – Negation Elimination ($\neg E$)

From a derivation of a proposition Q and its literal negation $\neg Q$ within a subproof involving an assumption $\neg(P)$, we can derive P out of the subproof.

$$
\begin{array}{lll}
n & \quad\vert\ \underline{\neg(P)} & \text{A} \\
\vdots & \quad\vert\ \vdots & \\
(n+1) & \quad\vert\ Q & \\
(n+2) & \quad\vert\ \neg Q & \\
(n+3) & P & \neg E,\ n\text{–}(n+2)
\end{array}
$$

Exercise 5:

1. $P \wedge \neg P \vdash R$
2. $(P \vee Q) \rightarrow R, P \vee Q, \neg R \vdash \neg W$

Derivation Rule – Disjunction Introduction ($\vee I$)

From P, we can derive $P \vee Q$ or $Q \vee P$.

$P \vdash P \vee Q$
$P \vdash Q \vee P$

Derivation Rule – Disjunction Elimination ($\vee E$)

From $P \vee Q$ and two derivations of R—one involving P as an assumption in a subproof, the other involving Q as an assumption in a subproof—we can derive R out of the subproof.

1	$P \vee Q$	P
n	$\quad\vert\ P$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ R$	
(i)	$\quad\vert\ Q$	A
$\vdots$	$\quad\vert\ \vdots$	
$(i+1)$	$\quad\vert\ R$	
(k)	R	$\vee E$, 1, n–$(n+1)$, (i)–$(i+1)$

Exercise 6:

1. $(P \vee Z) \rightarrow R, Z \vdash R \vee \neg L$
2. $P \vee Q, P \rightarrow R, Q \rightarrow R \vdash R$
3. $T \vee (Z \wedge M), (Z \wedge M) \rightarrow (\neg R \wedge S), T \rightarrow (\neg R \wedge S) \vdash \neg R$

Derivation Rule – Biconditional Introduction ($\leftrightarrow I$)

From a derivation of Q within a subproof involving an assumption P and from a derivation of P within a separate subproof involving an assumption Q, we can derive $P \leftrightarrow Q$ out of the subproof.

n	$\quad\vert\ P$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ Q$	
(i)	$\quad\vert\ Q$	A
$\vdots$	$\quad\vert\ \vdots$	
$(i+1)$	$\quad\vert\ P$	
(k)	$P \leftrightarrow Q$	$\leftrightarrow I$, n–$(n+1)$, (i)–$(i+1)$

Derivation Rule – Biconditional Elimination ($\leftrightarrow E$)

From $P \leftrightarrow Q$ and P, we can derive Q. And, from $P \leftrightarrow Q$ and Q, we can derive P.

$P \leftrightarrow Q, P \vdash Q$
$P \leftrightarrow Q, Q \vdash P$

Exercise 7:

1. $(F \vee Z) \rightarrow (T \wedge P), (P \vee M) \rightarrow (R \wedge Z) \vdash P \leftrightarrow Z$
2. $(P \vee \neg M) \leftrightarrow R, P \leftrightarrow (W \vee L), L \vdash R$

5.3 Proofs: Strategies

There are two main types of strategies: proof strategies and assumption strategies.

SP# 1 (E)	First, eliminate any conjunctions with $\wedge E$, disjunctions with $\vee E$, conditionals with $\rightarrow E$, and biconditionals with $\leftrightarrow E$. Then, if necessary, use any necessary introduction rules to reach the desired conclusion.
SP# 2 (B)	First, work backward from the conclusion using introduction rules (e.g. $\wedge I, \vee I, \rightarrow I, \leftrightarrow I$). Then, use SP# 1 (E).

Table 5.1 – Proof Strategies

Exercise 8:

1. $P \rightarrow (R \wedge M), (P \wedge S) \wedge Z \vdash R$
2. $P \rightarrow R, Z \rightarrow W, P \vdash R \vee W$

SA# 1 $(P, \neg Q)$	If the conclusion is an atomic proposition (or a negated proposition), assume the negation of the proposition (or the non-negated form of the negated proposition), derive a contradiction and then use $\neg I$ or $\neg E$.
SA# 2 $(\rightarrow)$	If the conclusion is a conditional, assume the antecedent, derive the consequent, and use $\rightarrow I$.
SA# 3 $(\wedge)$	If the conclusion is a conjunction, you will need two steps. First, assume the negation of one of the conjuncts, derive a contradiction, and then use $\neg I$ or $\neg E$. Second, in a separate subproof, assume the negation of the other conjunct, derive a contradiction, and then use $\neg I$ or $\neg E$. From this point, a use of $\wedge I$ will solve the proof.
SA#4 $(\vee)$	If the conclusion is a disjunction, assume the negation of the whole disjunction, derive a contradiction, and then use $\neg I$ or $\neg E$.

Table 5.2 – Assumption Strategies

Exercise 9:

1. $P \rightarrow Q, \neg Q \vdash \neg P$
2. $R \vdash \neg(D \vee L) \rightarrow R$

3. $\neg(P \vee R) \vdash \neg P \wedge \neg R$
4. $\neg(\neg P \wedge \neg Q) \vdash P \vee Q$

Consider $\neg(\neg P \wedge \neg Q) \vdash P \vee Q$. The strategy associated with assumptions is SA#3.

$$\begin{array}{lll}
1 & \neg(\neg P \wedge \neg Q) & P, P \vee Q \\
2 & \quad | \ \neg(P \vee Q) & A/P, \neg P \\
\vdots & \quad | \ \vdots & \\
\end{array}$$

The subgoal at this point is to generate a proposition P and its literal negation $\neg P$ in the subproof, but it is not clear how to do this. You cannot generate P and $\neg P$ out of nothing so consider what propositions you do have and try to derive a proposition that is a literal negation of these.

$$\begin{array}{lll}
1 & \neg(\neg P \wedge \neg Q) & P, P \vee Q \\
2 & \quad | \ \neg(P \vee Q) & A/P, \neg P \\
\vdots & \quad | \ \vdots & \\
\# & \quad | \ \neg P \wedge \neg Q \text{ or } P \vee Q & ? \\
\end{array}$$

We thus have two options:

Option # 1: derive $\neg P \wedge \neg Q$ since $\neg(\neg P \wedge \neg Q)$ is its literal negation Option # 2: derive $P \vee Q$ since $\neg(P \vee Q)$ is its literal negation

Consider option # 2. If we were to try to derive $P \vee Q$, we need to make an assumption, and the strategic rule associated with deriving disjunctions SA#4 says to assume the negation, derive P and $\neg P$, and then use $\neg E$ or $\neg I$. In the case of the above proof, the next step would be as follows:

$$\begin{array}{lll}
1 & \neg(\neg P \wedge \neg Q) & P, P \vee Q \\
2 & \quad | \ \neg(P \vee Q) & A/P, \neg P \\
3 & \quad | \quad | \ \neg(P \vee Q) & A/P, \neg P \\
\vdots & \quad | \quad | \ \vdots & \\
\# & \quad | \quad | \ \neg P \wedge \neg Q \text{ or } P \vee Q & ? \\
\end{array}$$

But this does not help since we still have no way to get P, $\neg P$ in the proof. So, consider option # 1. If we were to try and derive $\neg P \wedge \neg Q$, we would need to make an assumption, and the strategic rule associated with conjunctions SA# 3($\wedge$) says to assume the literal negation of each of the conjuncts in separate subproofs, derive P and $\neg P$ in each, and then use $\neg I$ or $\neg E$.

1	$\neg(\neg P \wedge \neg Q)$	$P, P \vee Q$
2	$\quad \neg(P \vee Q)$	$A/P, \neg P$
3	$\quad\quad P$	$A/P, \neg P$
$\vdots$	$\quad\quad \vdots$	
n	$\quad\quad Q$	$A/P, \neg P$
$\vdots$	$\quad\quad \vdots$	

Now the proof can be more easily solved.

Exercise 10:

1. $(\neg P \wedge L) \rightarrow \neg Q, (M \wedge T) \wedge (\neg R \wedge L), (M \wedge \neg R) \rightarrow (Z \wedge \neg P) \vdash \neg Q \vee (A \leftrightarrow B)$
2. $\neg R \vdash P \vee \neg W \rightarrow (Q \vee \neg R)$
3. $\vdash \neg(W \wedge \neg W)$
4. $P, (P \vee W) \rightarrow (R \wedge T), (T \vee \neg V) \leftrightarrow (\neg R \wedge T) \vdash S$
5. $\neg P \vee R \vdash P \rightarrow R$
6. $P \rightarrow R \vdash \neg P \vee R$

5.4 Proofs: Additional Derivation Rules (PD+)

The set of 10 intelim rules along with reiteration forms PD, a derivation system capable of proving any valid argument in PL. In other words, PD consists of all of the essential derivation rules we need. However, you may have noticed that the proofs for many straightforwardly valid arguments are overly difficult or time-consuming. For example, the proof of $P \vee Q, \neg Q \vdash P$ is overly complicated given that the argument is straightforwardly valid. In what follows, a number of additional derivation rules are added to PD to form PD+. These additional derivation rules serve to expedite the proof solving process.

Derivation Rule – Disjunctive Syllogism (DS)

From $P \vee Q$ and $\neg Q$, we can derive P. From $P \vee Q$ and $\neg P$, we can derive Q.

$P \vee Q, \neg Q \vdash P$
$P \vee Q, \neg P \vdash Q$

The general idea is that given a disjunction $P \vee Q$ and the literal negation of one of the disjuncts (either $\neg P$ or $\neg Q$), we can derive the other disjunct.

1	$P \vee (R \wedge S)$	P
2	$\neg(R \wedge S)$	P
3	P	DS, 1, 2

DERIVATION RULE – MODUS TOLLENS (MT)

From $P \to Q$ and $\neg Q$, we can derive $\neg P$.

$P \to Q, \neg Q \vdash \neg P$

The general idea is that given a conditional $P \to Q$ and the literal negation of the consequent $\neg Q$, the negation of the antecedent $\neg P$ can be derived.

1	$P \to (S \vee R)$	P
2	$\neg(S \vee R)$	P
3	$\neg P$	MT, 1, 2

DERIVATION RULE – HYPOTHETICAL SYLLOGISM (HS)

From $P \to Q$ and $Q \to R$, we can derive $P \to R$.

$P \to Q, Q \to R \vdash P \to R$

The idea is that if you have two conditionals $P \to Q$ and $Q \to R$ where the consequent of one conditional $P \to Q$ is the antecedent of the other conditional $Q \to R$, then you can derive a third conditional $P \to R$.

Exercise 11:

1. $(R \wedge T) \vee \neg W, S \wedge \neg\neg W \vdash R \wedge T$
2. $(P \wedge S) \to W, \neg W \wedge T \vdash \neg(P \wedge S)$
3. $(R \wedge T) \to \neg W, M \to (R \wedge T), \neg W \to (S \wedge R) \vdash M \to (S \wedge R)$
4. $P \vee \neg(R \vee S), R, L \to \neg P \vdash \neg L$

5.5 Proofs: Additional Derivation Rules (PD+), The Replacement Rules

All of the previous derivation rules have been inference rules, these are derivation rules that allow for deriving a proposition of one form from a proposition of another form. In addition to adding DS, MT, and HS to PD, we will also add a new kind of derivation rule, known as replacement rules. Replacement rules are derivation rules that allow for interchanging certain formulas or sub-formulas.

DERIVATION RULE – DOUBLE NEGATION (DN)

From P, we can derive $\neg\neg P$. From $\neg\neg P$, we can derive P.

$P \dashv\vdash \neg\neg P$

DN allows for replacing a single formula or single subformula with its doubly negated form or taking a doubly negated formula and replacing it with its unnegated form. For example,

1	$P \to R$	P,$\neg\neg(P \to R)$
2	$\neg\neg(P \to R)$	DN, 1

It is important to note that replacement rules can be applied to a single subformula. For example,

1	$P \vee \neg\neg(R \wedge S)$	$P/P \vee (R \wedge S)$
2	$P \vee (R \wedge S)$	DN, 1

But, be careful! DN must be applied to the whole of a formula or subformula and not to part of one subformula and part of another subformula:

1	$P \vee (R \wedge S)$	P
2	$P \vee \neg(\neg R \wedge S), NO!$	DN, 1

Derivation Rule – De Morgan's Laws (DeM)

From $\neg(P \vee Q)$, we can derive $\neg P \wedge \neg Q$. From $\neg P \wedge \neg Q$, we can derive $\neg(P \vee Q)$. From $\neg(P \wedge Q)$, we can derive $\neg P \vee \neg Q$. From $\neg P \vee \neg Q$, we can derive $\neg(P \wedge Q)$.

$\neg(P \vee Q) \dashv\vdash \neg P \wedge \neg Q$
$\neg(P \wedge Q) \dashv\vdash \neg P \vee \neg Q$

In the case of DeM, you can interchange a negated disjunction $\neg(P \vee Q)$ with a conjunction whose conjuncts are negated $\neg P \wedge \neg Q$ (and vice versa) and you can interchange a negated conjunction $\neg(P \vee Q)$ with a disjunction $\neg P \vee \neg Q$ whose disjuncts are negated (and vice versa).

For example, in the following proof, De Morgan's laws are applied to $\neg R \wedge \neg Q$ to derive $\neg(R \vee Q)$, i.e. turning a conjunction with negated conjuncts into a negated disjunction.

1	$P \to (R \vee Q)$	P
2	$\neg R \wedge \neg Q$	$P, \neg P$
3	$\neg(R \vee Q)$	DeM, 2
4	$\neg P$	MT, 1, 3

In the example below, DeM is applied to the negated disjunction $\neg(R \vee S)$ to derive a conjunction with two negated disjuncts.

1	$\neg(R \vee S)$	P
2	$\neg R \wedge \neg S$	DeM, 1
3	$\neg R$	$2 \wedge E$, 2

Derivation Rule – Implication (IMP)

From $P \to Q$, we can derive $\neg P \vee Q$. From $\neg P \vee Q$, we can derive $P \to Q$.
$P \to Q \dashv\vdash \neg P \vee Q$

In the case of IMP, you can interchange a negated conditional $P \to Q$ with a disjunction $\neg P \vee Q$.

Remembering that replacement rules can be applied to single sub-formula, notice how IMP is applied to the subformula '$P \to R$' in '$\neg(P \to R)$' in the following example:

1	$\neg(P \to R)$	P
2	$\neg(\neg P \vee R)$	IMP, 1
3	$\neg\neg P \wedge \neg R$	DeM, 2
4	$\neg\neg P$	$\wedge E$, 3
5	P	DN, 4

Exercise 12:

1. $\neg\neg P \to R, P, \neg\neg R \to (W \wedge Z) \vdash \neg\neg(W \wedge \neg\neg Z)$
2. $\neg(P \vee R) \to (\neg Z \vee \neg W), \neg P \wedge \neg R \vdash \neg(Z \wedge W)$
3. $(P \to R), (\neg P \vee R) \to (Z \to \neg R) \vdash \neg Z \vee \neg R$
4. $\neg P \vee R, \neg(P \to R) \vdash S$
5. $P \to \neg(Z \vee S), \neg(P \to R) \vdash \neg Z \vee W$
6. $P, \neg(\neg P \wedge \neg R) \to \neg(S \to T) \vdash S$

5.6 Proofs: Revised Strategic Rules

In enhancing our proof system from PD to $PD+$, we also want to enhance the strategies with which we solve proofs.

SP# 1 (E+) First, eliminate any conjunctions with $\wedge E$, disjunctions with DS or $\vee E$, conditionals with $\to E$ or MT, and biconditionals with $\leftrightarrow E$. Then, if necessary, use any necessary introduction rules to reach the desired conclusion.

SP# 2 (B) First, work backward from the conclusion using introduction rules (e.g. $\wedge I$, $\vee I$, $\to I$, $\leftrightarrow I$). Then, use SP# 1(E).

SP# 3 (EQ+) Use DeM on any negated disjunctions or negated conjunctions, and then use SP# 1(E). Use IMP on negated conditionals, then use DeM, and then use SP# 1(E).

Exercise 13:

1. $P \leftrightarrow (R \vee S), P \wedge \neg S, Q \rightarrow \neg R \vdash \neg Q$
2. $R \vee (M \wedge T), \neg R \wedge \neg W, L \rightarrow W \vdash \neg L$
3. $(R \vee M) \vee \neg(S \vee T), (S \vee T) \vee (Z \wedge E), \neg(R \vee M) \vdash E$
4. $\neg(P \vee R), \neg P \rightarrow \neg(M \vee S), \neg R \rightarrow \neg Q \vdash \neg M \wedge \neg Q$
5. $\neg(P \rightarrow R), P \rightarrow Z, \neg R \rightarrow M \vdash Z \wedge M$
6. $\neg(\neg P \rightarrow \neg R), Z \rightarrow P \vdash \neg Z \wedge R$
7. $\vdash \neg(P \rightarrow R) \rightarrow (S \rightarrow \neg R)$
8. $\vdash \neg(P \vee R) \rightarrow [(Z \rightarrow R) \rightarrow \neg Z]$
9. $\vdash [\neg(P \rightarrow M) \wedge \neg(T \rightarrow S)] \vee (P \vee \neg P)$

Derivation Rule – Conjunction Introduction $\wedge I$

$P, Q \vdash P \wedge Q$
$P, Q \vdash Q \wedge P$

Derivation Rule – Conjunction Elimination $(\wedge E)$

$P \wedge Q \vdash P$ or $P \wedge Q \vdash Q$

Derivation Rule – Conditional Introduction $(\rightarrow I)$

$$
\begin{array}{lll}
n & \quad | \; \underline{P} & \text{A} \\
\vdots & \quad | \; \vdots & \\
(n+1) & \quad | \; Q & \\
(n+2) & P \rightarrow Q & \rightarrow I,\ n\text{–}(n+1)
\end{array}
$$

Derivation Rule – Conditional Elimination $(\rightarrow E)$

$P \rightarrow Q, P \vdash Q$

Derivation Rule – Reiteration (R)

$P \vdash P$

Derivation Rule – Negation Introduction $(\neg I)$

$$
\begin{array}{lll}
n & \quad | \; \underline{P} & \text{A} \\
\vdots & \quad | \; \vdots & \\
(n+1) & \quad | \; Q & \\
(n+2) & \quad | \; \neg Q & \\
(n+3) & \neg(P) & \neg I,\ n\text{–}(n+2)
\end{array}
$$

Derivation Rule – Negation Elimination $(\neg E)$

n	$\quad\vert\ \neg(P)$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ Q$	
$(n+2)$	$\quad\vert\ \neg Q$	
$(n+3)$	P	$\neg E$, n–$(n+2)$

Derivation Rule – Disjunction Introduction ($\vee I$)

$P \vdash P \vee Q$ or $P \vdash Q \vee P$

Derivation Rule – Disjunction Elimination ($\vee E$)

1	$P \vee Q$	P
n	$\quad\vert\ P$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ R$	
(i)	$\quad\vert\ Q$	A
$\vdots$	$\quad\vert\ \vdots$	
$(i+1)$	$\quad\vert\ R$	
(k)	R	$\vee E$, 1, n–$(n+1)$, (i)–$(i+1)$

Derivation Rule – Biconditional Introduction ($\leftrightarrow I$)

n	$\quad\vert\ P$	A
$\vdots$	$\quad\vert\ \vdots$	
$(n+1)$	$\quad\vert\ Q$	
(i)	$\quad\vert\ Q$	A
$\vdots$	$\quad\vert\ \vdots$	
$(i+1)$	$\quad\vert\ P$	
(k)	$P \leftrightarrow Q$	$\leftrightarrow I$, n–$(n+1)$, (i)–$(i+1)$

Derivation Rule – Biconditional Elimination ($\leftrightarrow E$)

$P \leftrightarrow Q, P \vdash Q$ or $P \leftrightarrow Q, Q \vdash P$

Derivation Rule – Disjunctive Syllogism (DS)

$P \vee Q, \neg Q \vdash P$ or $P \vee Q, \neg P \vdash Q$

Derivation Rule – Modus Tollens (MT)

$P \rightarrow Q, \neg Q \vdash \neg P$

Derivation Rule – Hypothetical Syllogism (HS)

$P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R$

Derivation Rule – Double Negation (DN)

$P \dashv\vdash \neg\neg P$

Derivation Rule – De Morgan's Laws (DeM)

$\neg(P \vee Q) \dashv\vdash \neg P \wedge \neg Q$
$\neg(P \wedge Q) \dashv\vdash \neg P \vee \neg Q$

Derivation Rule – Implication (IMP)

$P \rightarrow Q \dashv\vdash \neg P \vee Q$